Characterization of A_8 by 3-Centralizers

A. A. Ligonnah[1*]

[1]Department of Mathematics, University of Botswana

Private Bag 0022, Gaborone, Botswana

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Abstract

Let G be a finite group containing a subgroup H isomorphic to an alternating group, A_n, such that G satisfies the 3-cycle property, namely 'for a 3-cycle $x \in H$, if $x^g \in H$ for any $g \in G$, then $g \in H$.'It is proved that for $n = 8$, G is isomorphic to LK, an extension of an elementary Abelian 2-group L by a group K isomorphic to either A_8 or $SL(5,2)$. If G is simple, it is shown that G is isomorphic to A_8 or $SL(5,2)$.

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1 Introduction

We investigate a finite group G which contains a subgroup H isomorphic to A_8, the alternating group on 8 letters, and has the 3-cycle property, namely:

Definition 1 (3-Cycle Property) *Let $x \in H$ be a 3-cycle. Then, if $x^g \in H$ for any $g \in G$, we have $g \in H$.*

*Corresponding author - E-mail: liggonah@mopipi.ub.bw

Just as the cases for $n = 5$ and 6 [9], if $\varphi : H \to A_8$ is the required isomorphism, we shall identify the elements of H with their images under φ; so that we can speak of an element of H being a 3-cycle.

We let $x \in H$ be the pre-image of (123) in A_8 in the isomorphism of H to A_8 and $y \in H$ be the pre-image of (456) in A_8. Then the 3-cycle property is equivalent to the hypotheses:
(i) The centralizer of x in G is contained in H;
(ii) The element x is not conjugate to the element xy in G.
The results for $n = 5$ and 6 have been given in earlier paper [9]. In this case for $n = 8$, the method of proof relies heavily on the Atlas of Finite Groups [2]. The case for $n = 7$ and G is simple has been proved in [10].
We prove the result in this paper by first proving when G is simple and then apply the general p-cycle property results, Proposition 1 in [9], to complete the proof.

2 Some Results on Group Classification

We give the results that we keep on referring in this paper for easy access.
Theorem 2.1[G.Mullineux, [**?**]]
Suppose $n \geq 9$ and A_n lies in the finite group G with the 3-Cycle Property. Then G possesses a normal elementary Abelian 2-subgroup X such that $X \bigcap A_n = 1$ and G is a semi-direct product $X \times A_n$.

Theorem 2.2[Feit-Thompson, [4]]
Let G be a finite non Abelian group with a self-centralizing subgroup of order 3. Then G is isomorphic to one of the following:

(i) MD_6, a semi-direct product, where M is a nilpotent group, D_6 is a dihedral group of order 6;

(ii) YA_5, a semi-direct product, where Y is an elementary Abelian 2-group;

(iii) $PSL(2,7)$.

We have the following corollary to the Feit-Thompson theorem given above:

Corollary 2.2
If G is a finite simple group with a self-centralizing element of order 3, then G is isomorphic to A_5 or $PSL(2, 7)$.

Theorem 2.3[Bryson, [3]]
Let G be a finite group with an element of order 3 whose centralizer is of order 9. Suppose also that G has an Abelian Sylow 3 -subgroup of order 9 and has two classes of elements of order 3. Then G is isomorphic to either A_6 or A_7.

Theorem 2.4[Proposition 4.1 [13]]
Let G be a finite group with a normal 2-subgroup Q such that G/Q is isomorphic to $PSL(2, 2^n)$, $n \geq 2$, and suppose an element of G of order 3 acts fixed-point-freely on Q. Then

(i) Q is elementary Abelian and is the direct product of minimal normal subgroups of G each of order 2^{2n}.

(ii) The Sylow 2-subgroup P of G is of class 2, and if $|Q| > 2^{2n}$, Q is the only Abelian subgroup of P of index 2^n.

Theorem 2.5[Proposition 4.2 in [13]]
Let G be a finite group, H a normal subgroup of G with G/H isomorphic to $PSL(2, 2^n)$, $n \geq 2$. Suppose an element t of G of order 3 acts fixed-point-freely on H. Then H is an elementary Abelian 2-group.

3 The Proof of the main Result

From now on, G will be a finite simple group containing a subgroup H isomorphic to A_8, and x in H is the pre-image of (123) in A_8, y in H is the pre-image of (567) in A_8 and conditions
(i) $C_G(x) < H$,
(ii) x is not conjugate to xy in G;
are satisfied.

We establish the following main result:

Theorem 3.1

Let G be a finite simple group containing a subgroup H isomorphic to A_8. Let x in H be the pre-image of (123) A_8 and y in H be the pre-image of (567). Assume

(i) $C_G(x) \leq H$;

(ii) x is not conjugate to xy in G.

Then G is isomorphic to A_8 or $SL(5,2)$.

Corollary 3.1

Let G be a finite group containing a subgroup $H \cong A_8$ such that G satisfies the 3-cycle property. Then G is an extension of an elementary Abelian 2-group F by a group L isomorphic to either A_8 or $SL(5,2)$.

Lemma 3.1

If $H_0 < H$, where $H \leq G$ with $H \simeq A_8$ and $x \in H_0$, then $N_G(H_0) < H$. If $P = \langle x, y \rangle$, then P is a Sylow 3-subgroup of G.

Proof

For any $g \in N_G(H_0)$, $x^g \in H$, so that x and x^g must be conjugate in H by hypothesis (ii) of the theorem. Thus $x^g = x^h$ for some $h \in H$, so that $x^{gh^{-1}} = x$. This implies that $gh^{-1} \in C_G(x) < H$ by hypothesis (i). Since $h \in H$, $g \in H$ also and hence $N_G(H_0) < H$. In particular, $N_G(P) < H$ so that P is a Sylow 3-subgroup of its own normalizer, hence it is a Sylow 3-subgroup of G.

4 A Survey of The Finite Simple Groups and Proof of the Theorem

We will make use of The Atlas of Finite Simple Groups, [2], that gives the list of finite simple groups and their properties that we can use to check which possibilities there are for G.

The finite simple groups are to be found among:

(1) The Cyclic Groups of Prime Order;

(2) The alternating group of degree at least five;

(3) The 26 Sporadic Simple Groups;

(4) The Chevalley groups.

Indeed, G is not cyclic so that (1) above does not hold. The finite simple group G we consider can be one of the alternating groups, one of the sporadic groups or one of the Chevalley groups. We have to consider each case at a time.

Proposition 2 *The finite simple group G is not one of the Sporadic groups.*

Proof. The 26 sporadic groups are listed below, from [2] page (viii), together with their corresponding orders:

LIST OF ALL SPORADIC GROUPS

Group	*Order*	*Investigator*
M_{11}	$2^4.3^2.5.11$	*Mathieu*
M_{12}	$2^6.3^3.5.11$	*Mathieu*
M_{22}	$2^7.3^2.5.7.11$	*Mathieu*
M_{23}	$2^7.3^2.5.7.11.23$	*Mathieu*
M_{24}	$2^{10}.3^3.5.7.11.23$	*Mathieu*
J_2	$2^7.3^3.5^2.7$	*Hall, Janko*
Suz	$2^{13}.3^7.5^2.7.11.13$	*Suzuki*
HS	$2^9.3^2.5^3.7.11$	*Higman, Sims*
McL	$2^7.3^6.5^3.7.11$	*McLaughlin*
Co_3	$2^{10}.3^7.5^3.7.11.23$	*Conway*
Co_2	$2^{18}.3^6.5^3.7^2.11.23$	*Conway*
Co_1	$2^{21}.3^9.5^4.7^2.11.13.23$	*Conway, Leech*
He	$2^{10}.3^3.5^2.7^3.17$	*Held/Higman, Mckay*
Fi_{22}	$2^{17}.3^9.5^2.7.11.13$	*Fischer*
Fi_{23}	$2^{18}.3^{13}.5^2.7.11.13.17.13$	*Fischer*
Fi_{24}	$2^{21}.3^{16}.5^2.7^3.11.13.17.23.29$	*Fischer*
HN	$2^{14}.3^6.5^6.7.11.19$	*Harada, Norton/Smith*
Th	$2^{15}.3^{10}.5^3.7^2.13.19.31$	*Thompson/Smith*
B	$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47$	*Fischer/Sims, Leon*
M	$2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.$	
	19.23.29.31.41.47..59.71	*Fischer*
J_1	$2^3.3.5.7.11.19$	*Janko*
$O'N$	$2^9.3^4.5.11.19.31$	*O'Nan*
J_3	$2^7.3^5.5.17.19$	*Janko/Higman, Mckay*
Ly	$2^8.3^7.5^6.7.11.31..37.67$	*Lyon/Sims*
Ru	$2^{14}.3^3.5^3..7.13.29$	*Rudvalis/Conway, Wales*
J_4	$2^{21}.3^3.5.7.11^3.23.29.31.37.43$	*Janko/Norton*

From the table, by considering the orders, the only groups with Sylow 3-subgroups of order 9 are M_{11}, M_{22}, M_{23} and HS. M_{11} is too small to contain H so is out. This leaves us with M_{22}, M_{23} and HS.

Considering maximal subgroups, M_{22} contains $L_3(4)$ $2^4 : A_6$ A_7 and 4 other groups of order less than $|A_7|$ as maximal subgroup as shown in [2] page 39. The order of $2^4 : A_6$ is 5760, orders of A_7 and the 4 other groups are too small for them to contain an isomorphic copy of A_8.The group $L_3(4)$ has order 2160 which equal to $|A_8|$. Let $S \in Syl_2(L_3(4))$.Then

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} | a,b,c \in GF(4) \right\}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ a' & 1 & 0 \\ c' & b' & 1 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ a+a' & 1 & 0 \\ c+c'+bb' & b+b' & 0 \end{pmatrix}$$

and

$$[A,B] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b'a-ba' & 0 & 1 \end{pmatrix} \in Z(S).$$

So $cl(S) = 2$.

On the other hand, $T = \langle (13)(24), (57)(68), (12)(56), (15)(26)(37)(48) \rangle$ is a Sylow 2-subgroup of A_8. Let $\sigma = (13)(24)$, $\tau = (12)(56)$ and $\varrho = (15)(26)(37)(48)$. Then $[\sigma, \tau] = (13)(24)(23)(14) = (12)(34)$ and $[\sigma, \tau, \varrho] = (12)(34)(56)(78)$, so $cl(T) \geq 3$. Hence $L_3(4) \ncong A_8$. This implies that M_{22} does not contain an isomorphic copy of A_8 as none of its maximal subgroups contains an isomorphic copy of A_8.

The group M_{23} contains A_8 as a maximal subgroup, but the elements of order 3 in M_{23} have only one conjugate class from [2], page 71. Hence the elements x and xy become fused in the bigger group M_{23}; so that M_{23} does not satisfy the hypothesis of the theorem.

The Higman-Sims group HS contains S_8 as a maximal subgroup, as is shown in [2] page 80, so HS contains A_8 inside S_8. This implies $C_G(x) \nsubseteq A_8$, hence HS does not satisfy the hypothesis put on G.

Hence, G is not a sporadic group.

Proposition 3 *The finite simple group G is not one of the Chevalley groups, other than $L_4(2) \cong A_8$ or $SL(5,2)$.*

Proof. The list and orders of the finite simple groups are given explicitly in pages 239 through 242 in [2]. The Chevalley groups are listed, and upon checking their orders, the only ones with orders divisible by $| A_8 |$, apart from A_8 and $SL(5,2)$, and have a Sylow 3-subgroup of order 9 are:

CHEVALLEY GROUPS OF ORDERS DIVISIBLE BY $|A_8|$, APART FROM A_8 AND $SL(5,2)$, WITH SYLOW-3 SUBGROUP OF ORDER 9.

Group	Order
$L_2(2^6)$	$2^6.3^2.5.7.13$
$S_4(7)$	$2^8.3^2.5^2.7^2$
$L_3(16)$	$2^{12}.3^2.5^2.7.13.17$
$S_4(13)$	$2^6.3^2.5.7^2.13^4.17$
$U_4(7)$	$2^{10}.3^2.5^2.7.11^6.19.61$
$L_4(11)$	$2^7.3^2.5^3.7.11^6.19.61$
$U_5(7)$	$2^{15}.3^2.5^2.7^{10}.11.19.43$

The group $L_2(2^6)$ has cyclic Sylow 3-subgroups of order 9, hence it does not satisfy the hypothesis that the group we consider has an elementary Abelian Sylow 3-subgroup P.

Every two elements in $SL_3(2^4)/Z(SL_3(2^4))$ of order 3 are conjugate, being similar to a diagonal matrix

$$\begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

where $\lambda \in GF(2^4)$ is a primitive root of unity.

Let $Q \in Syl_3(SL_3(2^4))$. Then Q is extra-special of exponent 3 and order 3^2, and Q is a conjugate of

$$\left\langle \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle. \tag{2}$$

So $L_3(2^4)$ has only one conjugate class of elements of order 3.

Let V be the 4-dimensional vector space over $GF(11)$. If $x \in SL_4(11)$ and $|x| = 3$, then either $\dim[V,x] = 2$ and $[V,x]$ is an irreducible $\langle x \rangle-$ module, or V is the sum of two 2-dimensional irreducible $\langle x \rangle$ -modules. In the first case,

$$C_{SL_4(11)}(x)$$

contains a subgroup of $SL_2(11)$. In the second case,

$$C_{GL_4(11)}(x) \cong GL_2(11^2).$$

In either case, centralizers of elements of order 3 in $SL_4(11)$ have orders divisible by 11. This carries over to $L_4(11)$, so that $L_4(11)$ does not satisfy the hypothesis that $C_G(x) \leq H$ which is isomorphic to A_8.

Let $G \in \{S_4(7), S_4(13), U_4(7), U_5(7)\}$. We know that, from [8], Theorem 8.8, $U_2(q) \cong L_2(q) \cong SL_2(q)$. Letting $q \in \{7, 13\}$, the groups $Sp_4(q)$, $SU_4(q)$ and $SU_5(q)$ respectively all have a subgroup $U \cong SL_2(q) \times SL_2(q)$ which contains a full Sylow 3-subgroup. This implies that $C_G(x)$ contains a subgroup of $L_2(q)$ whenever $x \in G$ and $|x| = 3$. This makes $C_G(x) \nleq H$ isomorphic to A_8, eliminating these groups. This completes the proof of the proposition.

The proof of the theorem is complete as we have shown that the finite simple group G satisfying the hypotheses of the theorem is either A_8 or $SL(5,2)$ by applying the results in The Atlas of Finite Simple Groups, [2].

Corollary 4 *Let G be a finite group containing a subgroup H with $H \cong A_8$ such that G satisfies the 3-cycle property. Then G is isomorphic $Q.L$, an extension of an elementary Abelian 2-group Q by a group L isomorphic to A_8 or $SL(5,2)$.*

Proof. Using the similar argument and results as in Corollary 6.2, with

$$\overline{G} = G/O_{3'}(G);$$

and $H \leq G$ with $H \cong A_8$. Then $\overline{H} \leq \overline{G}$. By Theorem 5.3, $O_{3'}(G) = F(G)$ and $\overline{G}$ is non-Abelian simple group. We have $\overline{H} \leq \overline{G}$, $C_G(\overline{x}) \leq \overline{H}$ and $\overline{x}$ is not conjugate to $\overline{xy}$ in $\overline{G}$. By Theorem 8.1, $\overline{G} \cong A_8$ or $\overline{G} \cong SL(5,2)$. This means that $G/F(G) \cong A_8$ or $G/F(G) \cong SL(5,2)$. Any element of order 3 in A_8 lies in a subgroup isomorphic to A_5, and A_5 acts on the group $F(G)$ with the element of order 3 acting fixed-point-freely. By Stewart's result, (Theorem 4.7), $F(G)$ is an elementary Abelian 2-group Q. Whence, $G/Q \cong A_8$ or $G/Q \cong SL(5,2)$. This implies $G \cong Q.A_8$ or $G \cong Q.SL(5,2)$, an extension of an elementary Abelian 2-group Q by either A_8 or $SL(5,2)$.

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