LOOP SPACE HOMOLOGY OF SOME HOMOGENEOUS SPACES

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ABSTRACT. Let X be a simply-connected compact oriented manifold of dimension m. We show that the Gerstenhaber structure of the loop space homology $\mathbb{H}_*(\mathrm{map}(S^1, X), \mathbb{Q})$ can be computed in terms of derivations on the minimal Sullivan model of X. The technique is applied to compute brackets in the loop space homology of homogeneous spaces.loop space homology.

Keywords: Derivations, Hochschild cohomology, Free loop space homology.

Mathematics Subject Classification (2010): Primary 55P62; Secondary 55M35

ISTJN 2013; 1(1):47-54.

1. INTRODUCTION

Let $A = \oplus_{n\geq 0} A^n$ be a commutative graded algebra over a commutative ring $\Bbbk$ and M a $\mathbb{Z}$-graded A-module. The A-tensor algebra $T_A(M)$ is defined by $T_A(M) = \oplus_{k\geq 0} T_A^k(M)$, where

$$T_A^k(M) = M \otimes_A M \otimes_A \cdots \otimes_A M \quad (k \geq 1 \text{ factors}) \text{ and } T_A^0(M) = A.$$

The exterior algebra $\wedge_A M$ is the commutative graded algebra obtained as the quotient of $T_A(M)$ by the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$, where $x, y \in T_A(M)$. The exterior product induces a graded commutative algebra structure on $\wedge_A M$.

Let $Z = \oplus_i Z_i$ be a $\mathbb{Z}$-graded free $\Bbbk$-module. There is a canonical isomorphism of commutative graded algebras

$$\varphi : \wedge_A(A \otimes Z) \to A \otimes \wedge_{\Bbbk} Z.$$

We assume that (A, d) is a differential graded algebra with a differential $d : A^n \to A^{n+1}$ and $A \otimes Z$ is an (A, d)-differential graded module, then $(\wedge_A(A \otimes Z), d)$ and $(A \otimes \wedge Z, d)$ are endowed with canonical differential graded algebra structures and φ becomes a homomorphism of differential graded algebras.

A derivation θ of degree r is a linear mapping $A^n \to A^{n-r}$ such that $\theta(ab) = \theta(a)b + (-1)^{r|a|} a\theta(b)$. Let $\mathscr{L}_i$ denote the vector space of all derivations of degree i and $\mathscr{L} = \oplus_i \mathscr{L}_i$. With the commutator bracket $\mathscr{L}$ becomes a graded Lie algebra. Using the grading convention $A^n = A_{-n}$, we may regard a derivation of degree r as increasing the lower degree by r. There is a differential $\delta : \mathscr{L}_n \to \mathscr{L}_{n-1}$ defined by $\delta\theta = [d, \theta]$.

Partial support from the Abdus Salam International Centre for Theoretical Physics and the International Mathematical Union is acknowledged.

Moreover $\mathscr{L}$ is a differential graded A-module with the action $(a\theta)(x) = a\theta(x)$. If $A = \wedge V$, we show that $\mathrm{Der} A \cong A \otimes V^{\#}$, where $V^{\#}$ is the graded dual of V (Lemma 3). With the above grading convention $V^{\#} = \oplus_{i\geq 1}(V^{\#})_i$ is positively graded.

On $L = s^{-1}\mathscr{L}$, we define a bracket of degree 1 by $\{\alpha,\beta\} = s^{-1}[s\alpha, s\beta]$ and a differential $\delta'(\alpha) = -\{d',\alpha\}$, where $d' = s^{-1}d \in L_{-2}$. We extend the bracket to $\wedge_A L = A \oplus L \oplus \wedge_A^2 L \oplus \dots$ by $\{a,b\} = 0$ for $a,b \in \wedge_A^0 L = A$, and $\{\alpha, a\} = -(-1)^{|\alpha|}(s\alpha)(a)$ for $\alpha \in L$. It is extended inductively on $\wedge_A^{\geq 2} L$ by forcing the Leibniz rule

$$\{\alpha, \beta\wedge\gamma\} = \{\alpha,\beta\}\wedge\gamma + (-1)^{(|\alpha|+1)|\beta|}\beta\wedge\{\alpha,\gamma\}. \tag{1}$$

Hence for α_i, β_i in L,

(2)

$$\{\alpha_1\wedge\cdots\wedge\alpha_m, \beta_1\wedge\cdots\wedge\beta_n\} = \sum_{i,j}(-1)^{\varepsilon_{ij}}\alpha_1\wedge\cdots\hat{\alpha}_i\wedge\alpha_m\wedge\{\alpha_i,\beta_j\}\wedge\cdots\wedge\hat{\beta}_j\wedge\cdots\beta_n,$$

where $\hat{}$ means omitted and $\varepsilon_{ij} = \sum_{k>i}|\alpha_k||\alpha_i| + \sum_{k<j}|\beta_k||\beta_j|$. The above bracket (called Nijenhuis-Schouten bracket) turns $\wedge_A L$ into a Gerstenhaber algebra. See [15, §2] for instance.

The differential δ' extends into an algebra differential d_0 on $\wedge_A L$ by the same rule, that is, $d_0\alpha = -\{d',\alpha\}$, for $\alpha \in \wedge_A L$. It comes from the Leibniz rule (1) that d_0 is a derivation. Moreover the Jacobi identity ensures that d_0 is compatible with the bracket. Hence $(\wedge_A L, d_0)$ becomes a differential graded Gerstenhaber algebra.

Let $A = (\wedge V, d)$ be a Sullivan algebra and $Z = s^{-1}V^{\#}$. In the same way one extends the bracket and the differential of $L = A\otimes Z$ to $A\otimes\wedge Z$. The main result states.

Theorem 1. *Let $A = (\wedge V, d)$ be a Sullivan algebra over a field $\Bbbk$ of characteristic 0, where $V = \oplus_{i\geq 1}V^i$ and L the desuspension of $\mathscr{L} = \mathrm{Der}\wedge V$ with the desuspended differential and $Z = s^{-1}V^{\#}$. Then $\varphi : (\wedge_A L, d_0) \to (A\otimes\wedge Z, D)$ extends to an isomorphism of differential graded Gerstenhaber algebras.*

Let $\bar{A}$ be the kernel of the augmentation $\varepsilon : A \to \Bbbk$. We denote by $C^*(A;A) = \mathrm{Hom}(T(s\bar{A}), A)$ (resp. $HH^*(A;A)$) the Hochschild complex (resp. cohomology) of the cochain algebra A with coefficients in A [13]. We recall the following result which is a combination of Theorems B and C in [12].

Theorem 2. *If $A = (\wedge V, d)$ is a Sullivan algebra, then there is a mapping ϕ : $(\wedge_A L, d_0) \to C^*(A;A)$ which induces an isomorphism of graded Gerstenhaber algebras in homology.*

Note that ϕ does not induce a bijective map in homology if A is not a Sullivan algebra [2, Theorem 6.2].

By combining Theorems 1 and 2, we get an easy method to compute the Gerstenhaber bracket on $HH^*(A;A)$, when A is a Sullivan algebra.

2. Lie algebra of derivations over a Sullivan algebra

From now on $\Bbbk$ is a field of characteristic 0. We will rely on Sullivan models theory in rational homotopy, of which details can be found in [17, 19, 7]. A Sullivan algebra is a commutative differential graded algebra (cdga for short) of the form $(\wedge V, d)$, where $V = \bigcup_{k\geq 0} V(k)$, and $V(0) \subset V(1) \ldots$, such that $dV(0) = 0$ and $dV(k) \subset \wedge V(k-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$.

If (A,d) is a cdga of which the cohomology is connected and finite in each degree, then there is a Sullivan algebra $(\wedge V, d)$ equipped with a quasi-isomorphism $(\wedge V, d) \to (A,d)$. For each simply connected space, Sullivan defines a cdga $A_{PL}(X)$ that uniquely determines the rational homotopy type of X [17]. A Sullivan (minimal) model of a simply connected space X is a Sullivan (minimal) model of $A_{PL}(X)$.

Let $A = (\wedge V, d)$ be the Sullivan minimal model of a simply connected space X. We assume that each V^i is a finite dimensional vector space.

Let $\mathscr{L} = \mathrm{Der} \wedge V$. With the grading $A_{-n} = A^n$, if $\theta \in \mathscr{L}_k$ and $a \in A^i$, then $a\theta \in \mathscr{L}_{k-i}$. Moreover, if $\theta, \theta' \in \mathscr{L}$ and $a \in A$, then

$$\begin{aligned} [\theta, a\theta'](x) &= \theta(a\theta'(x)) + (-1)^{|\theta||a\theta'|}(a\theta')(\theta(x)) \\ &= \theta(a)\theta'(x) + (-1)^{|\theta||a|}a(\theta\theta')(x) + (-1)^{|\theta||a\theta'|}a(\theta'\theta)(x) \\ &= \theta(a)\theta'(x) + (-1)^{|\theta||a|}a[\theta,\theta'](x). \end{aligned}$$

Hence

$$[\theta, a\theta'] = \theta(a)\theta' + (-1)^{|\theta||a|}a[\theta,\theta']. \tag{3}$$

Lemma 3. *Let $(v_i)_{i\in I}$ be a homogeneous linear basis of V and, for $i \in I$, let θ_i be the derivation of $\wedge V$ uniquely determined by*

$$\theta_i(v_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

The graded $\wedge V$-module $\mathscr{L} = \mathrm{Der} \wedge V$ is freely generated by the derivations θ_i ($i \in I$).

Proof. Let us denote by $V^\#$ the graded dual of V. The restriction of each θ_i to V is an element of $V^\#$ of upper degree $-|v_i|$. Thus we have an isomorphism of graded $\wedge V$-modules

$$\mathrm{Der} \wedge V \cong \mathrm{Hom}(V, \wedge V) \cong (\wedge V) \otimes V^\#$$

□

The derivation θ_i will be denoted by $(v_i, 1)$.

However $H_*(\mathscr{L})$ is not a free $H^*(\wedge V, d)$-module as in the following example.

Example 4. Consider the commutative differential graded algebra $\wedge(x_2, y_5)$, with $dx_2 = 0$, $dy_5 = x_2^3$. As an A-module, $\mathscr{L}$ is spanned by $\{\alpha_2, \alpha_5\}$ where $\alpha_2 = (x_2, 1)$ and $\alpha_5 = (y_5, 1)$. Moreover $\delta\alpha_5 = 0$ and $\delta\alpha_2 = -3x_2^2\alpha_5$. Therefore $[x_2^2][\alpha_5] = 0$ in $H_*(\mathscr{L})$.

Proof of Theorem 1. The isomorphism $\mathrm{Der}\wedge V \cong (\wedge V) \otimes V^\#$ extends to an isomorphism of graded algebras

$$\varphi : \wedge_{\wedge V} s^{-1}(\mathrm{Der}\wedge V) \cong \wedge_{\wedge V} s^{-1}(\wedge V \otimes V^\#) \cong \wedge V \otimes \wedge(s^{-1}V^\#)$$

The next two lemmas will complete the proof.

Lemma 5. *The map φ is compatible with brackets.*

Proof. We consider $\alpha, \beta_1, \beta_2 \in s^{-1}(\mathrm{Der}\wedge V)$.

$$\begin{aligned}
\varphi(\{\alpha, \beta_1\beta_2\}) &= \varphi(\{\alpha, \beta_1\}\beta_2) + (-1)^{(|\alpha|+1)|\beta_1|}\varphi(\beta_1\{\alpha, \beta_2\}) \\
&= \varphi(\{\alpha, \beta_1\})\varphi(\beta_2) + (-1)^{(|\alpha|+1)|\beta_1|}\varphi(\beta_1)\varphi(\{\alpha, \beta_2\}) \\
&= \{\varphi(\alpha), \varphi(\beta_1)\}\varphi(\beta_2) + (-1)^{(|\alpha|+1)|\beta_1|}\varphi(\beta_1)\{\varphi(\alpha), \varphi(\beta_2)\} \\
&= \{\varphi(\alpha), \varphi(\beta_1)\varphi(\beta_2)\} \\
&= \{\varphi(\alpha), \varphi(\beta_1\beta_2)\}.
\end{aligned}$$

By an induction argument, one deduces that φ respects the brackets. □

Lemma 6. *The map $\varphi : (\wedge_{\wedge V} s^{-1}(\mathrm{Der}\wedge V), d_0) \to (\wedge V \otimes \wedge(s^{-1}V^\#), D)$ commutes with differentials.*

Proof. The differential D on $\wedge V \otimes \wedge(s^{-1}V^\#)$ is defined by $D\alpha = -\{\varphi(d'), \alpha\}$, where $d' = s^{-1}d$. As φ is compatible with brackets, we deduce that

$$\varphi(d_0\alpha) = -\varphi(\{d', \alpha\}) = -\{\varphi(d'), \varphi(\alpha)\} = D(\varphi(\alpha)).$$

Hence φ commutes with differentials. □

Remark 7. Let $Z = s^{-1}V^\#$. The $\wedge V$-module $(\wedge V \otimes \wedge Z, D)$ is the "dual" of the Sullivan model of LX described by Sullivan and Vigué-Poirrier [20]. However the former carries the Gerstenhaber structure of the free loop space homology as developed in the sequel.

3. The free loop space homology spectral sequence

We apply the above result in the computation of the free loop space homology. Let X be a closed oriented manifold of dimension m and $LX = \mathrm{map}(S^1, X)$ the space of free loops on X. The loop homology of X is the homology of LX with a shift of degrees, $\mathbb{H}_*(LX) = H_{*+m}(LX)$ and an associative and graded commutative product

$$\mu : \mathbb{H}_p(LX) \otimes \mathbb{H}_q(LX) \to \mathbb{H}_{p+q}(LX)$$

called *loop product* [3]. When coefficients are taken in a field there is an isomorphism of graded vector spaces [14]

$$HH_*(C^*X; C^*X) \cong H^*(LX)$$

which dualizes in

$$HH^*(C^*X; C_*X) \cong H_*(LX).$$

If $\mathbb{k}$ is of characteristic 0 and X is simply connected, there is an isomorphism of Gerstenhaber algebras [10, 11, 9]

$$\Phi : \mathbb{H}_*(LX) \to HH^*(C^*X;C^*X).$$

Moreover given a Sullivan minimal model $A = (\wedge V, d)$ of X, one has an isomorphism of Gerstenhaber algebras [8, Proposition 3.3]

$$HH^*(A;A) \cong HH^*(C^*X;C^*X).$$

Hence we have isomorphisms of Gerstenhaber algebras

$$\mathbb{H}_*(LX,\mathbb{Q}) \xrightarrow{\cong} HH^*(A;A) \xleftarrow{\cong} H_*(\wedge_A L, d_0) \xrightarrow{\cong} H_*(\wedge V \otimes \wedge Z, d),$$

where $L = s^{-1}(\mathrm{Der}\wedge V)$. In this section we describe a spectral sequence of $\wedge V \otimes \wedge Z$ that simplifies the computation of $\mathbb{H}_*(LX,\mathbb{Q})$ in some cases.

We recall the following definitions [1, 6].

Definition 8. Let (A,d) be a differential graded algebra. A differential graded module (M,d) over (A,d) is called free if it is free as an A-module and the basis is made up of cycles.

For instance, if $A = (\wedge V, d)$ is a Sullivan algebra then $\mathscr{L} = \mathrm{Der}(\wedge V, d)$ is not necessarily free as a differential graded module, although it is free as an A-module (Lemma 3).

Definition 9. A differential (A,d)-module (M,d) is called semifree if there is a filtration $F_0M \subset F_1M \subset \cdots \subset M$ such that each $F_iM/(F_{i-1}M)$ is free on a basis of cycles.

If X is an n-stage Postnikov tower, then X admits a Sullivan algebra of the form $(\wedge(V_1 \oplus \cdots \oplus V_n), d)$, where $dV_1 = 0$ and $dV_i \subset \wedge(V_1 \oplus \cdots \oplus V_{i-1})$.

Proposition 10. *Under the assumptions above,* $\mathscr{L} = \mathrm{Der}(\wedge V, d)$ *is semifree over* $(\wedge V, d)$.

Proof. Define a filtration on the Lie algebra of derivations as follows.

$$F_p\mathscr{L} = \{\theta \in \mathscr{L} : \theta(V_0 \oplus \cdots \oplus V_{n-p-1}) = 0\}.$$

We get a filtration $0 \subset F_0\mathscr{L} \subset F_1\mathscr{L} \subset \cdots \subset F_{n-1}\mathscr{L} \subset F_n\mathscr{L} = \mathscr{L}$. For instance if $V_n =< v_{n,1}, \dots, v_{n,k} >$ then $F_0\mathscr{L} = Z^0$ is spanned by $\{\theta_{0,1}, \dots, \theta_{0,k}\}$ where $\theta_{0,i} = (v_{n,i}, 1)$. Assume $V_{n-1} =< v_{n-1,1}, \dots, v_{n-1,l} >$, then $F_1\mathscr{L}/F_0\mathscr{L} = Z^1$ is spanned by derivations $\{\theta_{1,1}, \dots, \theta_{1,l}\}$ where $\theta_{1,j} = (v_{n-1,j}, 1)$. Moreover $\delta Z^1 \subset (\wedge V) \otimes Z^0 = F_0\mathscr{L}$. In general, $F_k\mathscr{L}/F_{k-1}\mathscr{L} = Z^k$ is spanned by derivations $\{\theta_{k,1}, \dots, \}$ where $\theta_{k,i} = (v_{n-k,i}, 1)$ and $\delta Z^k \subset (\wedge V) \otimes (Z^0 \oplus \cdots \oplus Z^{k-1})$. This defines a semifree filtration of $\mathscr{L}$, hence $(\mathscr{L}, \delta)$ is a semifree differential module over $(\wedge V, d)$. □

It comes from the definition that $[F_p\mathscr{L}, F_q\mathscr{L}] \subset F_r\mathscr{L}$, where $r = \max\{p,q\}$. Hence $[F_p\mathscr{L}, F_q\mathscr{L}] \subset F_{p+q}\mathscr{L}$. The filtration induces a spectral sequence of differential Lie algebras such that $E^0_{m,*} = F_m\mathscr{L}/F_{m-1}\mathscr{L} \cong A \otimes Z^{m,*}$ and $d_0 = d_A \otimes 1$.

Hence $E^1_{m,*} \cong H(A) \otimes Z^m$. The E^1-term, together with differentials, yields

$$\begin{array}{ccccc} E^1_{n,*} & \xrightarrow{d_1} & E^1_{n-1,*} \cdots & \xrightarrow{d_1} & E^1_{0,*} \\ \| & & \| & & \| \\ H(A)\otimes Z^n_* & \xrightarrow{d_1} & H(A)\otimes Z^{n-1}_* \cdots & \xrightarrow{d_1} & H(A)\otimes Z^0_*. \end{array}$$

In particular if $(\wedge V, d) = (\wedge(V_0 \oplus V_1), d)$ with $dV_0 = 0$ and $dV_1 \subset \wedge V_0$, then the above spectral sequence collapses at E^2-level.

Example 11. Consider the cdga $(\wedge(x,y),d)$ with $|x| = 2$, $|y| = 5$ and $dy = x^3$. Here $H = (\wedge x)/(x^3)$ and Z^0 (resp. Z^1) is spanned by $z_0 = (y,1)$ (resp. $z_1 = (x,1)$). Hence $E^1 = H \otimes Z$. Moreover $d_1 z_0 = 0$, $d_1 z_1 = x^2 z_0$ and $d_1(xz_1) = 0$. Therefore the E^2-term is spanned by $\{z_0, xz_0, xz_1, x^2 z_1\}$ as a vector space. We note that xz_1 and $x^2 z_1$ are of respective degrees 0 and -2.

More generally, let X be a 2-stage Postnikov tower of which the Sullivan minimal model is given by $(\wedge(V_0 \oplus V_1), d)$ where V_0 is even, V_1 odd, $dV_1 \subset \wedge V_0$ and $H^*(\wedge(V_0 \oplus V_1), d) \cong \wedge V_0/(dV_1)$. If X verifies the Halperin conjecture, then $H_*(\mathrm{Der} \wedge V)$ is concentrated in odd degrees [16], that is, $d_1 : H \otimes Z^1 \to H \otimes Z^0$ is injective in positive degrees, or equivalently, $\ker d_1$ is concentrated in non positive degrees.

Let X be a simply connected compact oriented m-manifold of which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional. Then X is a n-stage Postnikov tower and its Sullivan minimal model is given by $A = (\wedge(V_0 \oplus \cdots \oplus V_n), d)$, where $dV_i \subset \wedge(V_0 \oplus \cdots \oplus V_{i-1})$. Let $Z = s^{-1}V^{\#}$ and $Z^k = s^{-1}V^{\#}_{n-k}$. We define a filtration on $A \otimes \wedge Z$ by $F_p = A \otimes \wedge(Z^0 \oplus \cdots \oplus Z^{p-1})$. As $F_p F_q \subset F_r$, where $r = \max\{p,q\}$, hence $F_p F_q \subset F_{p+q}$. Moreover $\{F_p, F_q\} \subset F_s$, where $s = \max\{p,q\}$, therefore $\{F_p, F_q\} \subset F_{p+q}$.

This filtration yields a spectral sequence of Gerstenhaber algebras for which $E^1 = H^*(A) \otimes \wedge Z$ and which converges to $H_*(A \otimes \wedge Z, d) \cong \mathbb{H}_*(LX, \mathbb{Q})$.

Proposition 12. *If $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then $s\mathbb{H}_*(LX, \mathbb{Q})$ contains an abelian polynomial sub Lie algebra.*

Proof. Let $A = (\wedge V, d)$ be the Sullivan minimal model of X. Consider the above filtration of $\wedge V \otimes \wedge Z$ for X. As $dZ^0 = 0$ and $dZ \subset A^+ \otimes Z$, then $\wedge Z^0 \subset H_*(A \otimes \wedge Z)$. As $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then Z^0 contains a subspace spanned by $\{\alpha_1, \ldots, \alpha_k\}$, and the suspension of each α_i corresponds to a non zero rational Gottlieb element of X. As X is a finite dimensional manifold, each α_i is even [5]. Moreover $\{\alpha_i, \alpha_j\} = 0$, hence the suspension of $\wedge(\alpha_1, \ldots, \alpha_k)$ is an abelian sub Lie algebra. □

4. Computations for homogeneous spaces

In [18] Tamanoi determines among other things the Lie bracket of the integral free loop space homology of complex Stiefel manifolds. We restrict here to the computation of the rational free loop space homology of a simply connected homogeneous space X. In this case the Sullivan minimal model of X is of the form $A = (\wedge(x_1,\dots,x_n,y_1,\dots,y_m),d)$ where $|x_i|$ is even, $|y_i|$ is odd, $dx_i = 0$ and $dy_i = f_i \in \wedge(x_1,\dots,x_n)$. The differential d_1 on $H^*(A)\otimes\wedge Z$ can be made explicit. As Z^0 and Z^1 are spanned by $\{s^{-1}(y_j,1), 1\le j\le m\}$ and $\{s^{-1}(x_i,1), 1\le i\le n\}$ respectively, therefore Z^0 (resp. Z^1) is concentrated in even (resp. odd) degrees. Moreover $dZ^0 = 0$, and if $z_{1,i} = s^{-1}(x_i,1)$ and $z_{0,j} = s^{-1}(y_j,1)$ then $dz_{1,i} = \sum_j \frac{\partial f_j}{\partial x_i} z_{0,j}$. We have then proved the following result.

Proposition 13. *Let X be a homogeneous space and $B = H^*(X,\mathbb{Q})$. Then $\mathbb{H}_*(LX,\mathbb{Q}) \cong H_*(B\otimes\wedge(Z^0\oplus Z^1),d_1)$, where $d_1Z^0 = 0$ and $d_1Z^1 \subset B^+\otimes Z^0$.*

We can therefore apply the just proved Proposition 13 to compute the loop space homology of homogeneous spaces. We give here two examples.

Example 14. [4, 8, 11] Consider $X = \mathbb{C}P(n)$ of which the Sullivan minimal model is $(\wedge(x,y),d)$, $dx = 0$, $dy = x^{n+1}$. Therefore

$$\mathbb{H}_*(\mathbb{C}P(n),\mathbb{Q}) \cong H_*(\wedge x/(x^{n+1})\otimes\wedge(z_1,z_{2n}),d),\; dz_{2n} = 0,\; dz_1 = (n+1)x^n z_{2n}.$$

Here z_1 and z_{2n} are of respective degrees 1 and $2n$. Homology classes are

$$\{x^j z_{2n}^k,\, x^i z_1,\, x^i z_1 z_{2n}^k,\quad k\ge 0,\quad 0\le j\le n-1,\quad 1\le i\le n\}.$$

Brackets can be computed from the Lie algebra structure of derivations on $(\wedge(x,y),d)$. For instance $\{x^i z_{2n}, x^j z_{2n}\} = 0$, $\{x^i z_1, x^j z_{2n}\} = jx^{i+j-1}z_{2n}$, $\{xz_1x^iz_{2n}^k, xz_1x^jz_{2n}^l\} = (i-j)xz_1x^{i+j}z_{2n}^{k+l}$. In particular $\{xz_1, x^jz_{2n}\} = jx^jz_{2n}$, hence $\mathrm{ad}^k(xz_1)\neq 0$, for $k\ge 1$.

Example 15. We consider the Sullivan minimal model of $X = Sp(5)/SU(5)$ which is given by $A = (\wedge(x_6,x_{10},y_{11},y_{15},y_{19},d)$ with $dx_i = 0$, $dy_{11} = x_6^2$, $dy_{15} = x_6x_{10}$, $dy_{19} = x_{10}^2$. The rational cohomology is given by classes of $\{1, x_6, x_{10}, x_6y_{15}-x_{10}y_{11}, x_{10}y_{15}-x_6y_{19}, x_6(x_{10}y_{15}-x_6y_{19})\}$.

The loop space homology is computed from the complex

$$\begin{aligned}(A\otimes\wedge(z_{10},z_{14},z_{18},w_5,w_9),d),\quad & dz_i = 0, dw_5 = 2x_6z_{10}+x_{10}z_{14},\\ & dw_9 = x_6z_{14}+2x_{10}z_{18}.\end{aligned}$$

It contains $H^*(X)\otimes\wedge(z_{10},z_{14},z_{18})/I$ where I is the ideal generated by $\{dw_5, dw_9\}$, but also x_6w_i and $x_{10}w_i$. Non zero brackets include

$$\begin{aligned}&\{x_6w_5, x_6z_i^k\} = x_6z_i^k,\quad \{x_6w_9, x_{10}z_i^k\} = x_6z_i^k,\\ &\{x_{10}w_5, x_6z_i^k\} = x_{10}z_i^k,\quad \{z_{10},(x_6y_{15}-x_{10}y_{11})z_i^k\} = -x_{10}z_i^k,\\ &\{z_{14},(x_6y_{15}-x_{10}y_{11})z_i^k\} = x_6z_i^k,\quad \{z_{18},(x_{10}y_{15}-x_6y_{19})z_i^k\} = -x_6z_i^k.\end{aligned}$$

Hence, for $\alpha = x_6w_5$, $\mathrm{ad}^k\alpha\neq 0$, $k\ge 1$. It is the same for $\beta = x_{10}w_9$.

Acknowledgement. I would like to thank the *Université de Lille I*, especially Prof D. Tanré for their warm hospitality during the initial stage of the preparation of this paper.

REFERENCES

[1] L. Avramov and S. Halperin, *Through the looking glass: a dictionary between rational homotopy and local algebra*, Algebra, Algebraic Topology and their interactions (J.E. Ross, ed.), Lecture Notes in Mathematics, vol. 1183, Springer, Berlin, 1986, pp. 1–31.

[2] A. Cattaneo, D. Fiorenza, and R. Longoni, *On the Hochschild-Kostant-Rosenberg map for graded manifolds*, Int. Math. Res. Not. 2005, **62** (2005), 3899–3918.

[3] M. Chas and D. Sullivan, *String topology*, preprint math GT/9911159, 1999.

[4] R.L. Cohen, J.D.S Jones, and J. Yuan, *The loop homology algebra of spheres and projective spaces*, Categorical decompositions techniques in algebraic topology (G. Aron, J. Hubbuck, R. Levi, and M. Weiss, eds.), Progress in Mathematics, vol. 215, Birkhäuser, Basel, 2003, pp. 77–92.

[5] Y. Félix and S. Halperin, *Rational LS category and its applications*, Trans. Amer. Math. Soc. **273** (1982), 1–38.

[6] Y. Félix, S. Halperin, and J.-C. Thomas, *Differential graded algebras in topology*, Handbook of Algebraic Topology (I.M. James, ed.), North-Holland, 1995, pp. 829–865.

[7] ———, *Rational homotopy theory*, Graduate Texts in Mathematics, no. 205, Springer-Verlag, New-York, 2001.

[8] Y. Félix, L. Menichi, and J.-C. Thomas, *Gerstenhaber duality in Hochschild cohomology*, J. of Pure and Applied Algebra **199** (2005), 43–59.

[9] Y. Félix and J.-C. Thomas, *Rational BV-algebra in string topology*, Bull. Soc. Math. France **136** (2008), 311–327.

[10] Y. Félix, J.-C. Thomas, and M. Vigué, *The Hochschild cohomology of a closed manifold*, Publ. Math. IHES. **99** (2004), 235–252.

[11] ———, *Rational string topology*, J. Eur. Math. Soc. (JEMS) **9** (2008), 123–156.

[12] J.-B. Gatsinzi, *Derivations, Hochschild cohomology and the Gottlieb group*, Homotopy Theory of Function Spaces and Related Topics (Y. Félix, G. Lupton, and S. Smith, eds.), Contemporary Mathematics, vol. 519, American Mathematical Society, Providence, 2010, pp. 93–104.

[13] M. Gerstenhaber, *The cohomology structure of an associative ring*, Annals of Math. **78** (1963), 267–288.

[14] J. D. S. Jones, *Cyclic homology and equivariant homology*, Inv. Math. **87** (1987), 403–423.

[15] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque (Numéro Hors Série) (1985), 257–271.

[16] W. Meier, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1982), 329–340.

[17] D. Sullivan, *Infinitesimal computations in topology*, Publ. I.H.E.S. **47** (1977), 269–331.

[18] H. Tamanoi, *Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds*, Int. Math. Res. Not. (2006), 23pp.

[19] D. Tanré, *Homotopie Rationnelle: Modèles de Chern, Quillen, Sullivan*, Lecture Notes in Mathematics, no. 1025, Springer, Berlin, 1983.

[20] M. Vigué-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geom. **11** (1976), 633–644.

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