Characterization of A_n for $n = 5, 6$ by 3-centralizers

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Abstract

Let G be a finite group containing a subgroup H isomorphic to an alternating group, A_n, such that G satisfies the 3-cycle property, namely 'for a 3-cycle $x \in H$, if $x^g \in H$ for any $g \in G$, then $g \in H$.' It is proved that G is isomorphic to LK, an extension of an Abelian 2-group L by a group K isomorphic to either A_5 for $n = 5$; or A_6 or A_7 for $n = 6$. If G is simple, we establish that G is isomorphic to A_5 for $n = 5$; or G is isomorphic to A_6 or A_7 for $n = 6$.

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1 Introduction

We investigate a finite group G which contains a subgroup H isomorphic to A_n, the alternating group on n letters, and has the 3-cycle property:

Definition 1 [3-Cycle Property]
Let $x \in H$ be a 3-cycle. Then, if $x^g \in H$ for any $g \in G$, we have $g \in H$.

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If $\varphi : H \to A_n$ is the required isomorphism, we shall identify the elements of H with their images under φ so that we can speak of an element of H being a 3-cycle.

We let $x \in H$ be the pre-image of (123) in A_n in the isomorphism of H to A_n and $y \in H$ be the pre-image of (456) in A_n, for $n \geq 6$. If $g \in C_G(x)$, then $x = x^g \in H$ so that by the 3-cycle property, $g \in H$. Hence $C_G(x) \leq H$. Suppose x is conjugate to xy in G. Then $x^g = xy$ for some $g \in G$. It follows that $x^g \in H$. By the 3-cycle property, $g \in H$ so that x is conjugate to xy in H, a contradiction as H is isomorphic to A_n, x is the pre-image of (123), xy is the pre-image of (123)(456) which are not conjugate in A_n.

Therefore, by the above reason, the 3-cycle property is equivalent to the hypotheses:
(i) The centralizer of x in G is contained in H;
(ii) The element x is not conjugate to the element xy in G.
We consider the cases $n = 5, 6, 7$ and 8. Mullineux [6] considered the cases $n \geq 9$ and proved that if G is not simple, then $G \simeq XH$, a semi-direct product, where $X \bigcap H = 1$ and X is an elementary Abelian normal 2-group. If G is simple, then $G \simeq A_n$.

For the proof of results for $n \geq 9$, Mullineux used a 3-cycle. In our case, we shall use a 3-cycle in order to complete the characterisation. We feel this is an important consideration as it fills the gap for the cases $n = 5$, 6.

A more general p-cycle property has been considered by relaxing the conditions on the group G and the prime p. The following Proposition is established.

Proposition 1
Let G be a finite group containing a subgroup H such that H is a non-Abelian simple group, and H has an element x of prime order p with $C_G(x) \leq H$. Then $O_{p'}(G) = F(G)$ and $G/O_{p'}(G)$ is non-Abelian simple; where $F(G)$ is the Fitting subgroup of G.

From Proposition 1 and its proof, the finite group having the 3-cycle property must have a normal series $1 \trianglelefteq O_{3'}(G) \triangleleft G$ with $O_{3'}(G)$ nilpotent and $G/O_{3'}(G)$ non-Abelian and simple. It follows from the proof of Proposition 1 that $G/O_{3'}(G)$ is a simple group which has the 3-cycle property. We therefore classify the finite simple groups with the 3-cycle property listed.

2 Some Results on Group Classification

In this paper, we present results that we will keep on referring to, in different stages of the proofs of the main results of this paper. We present these results in the order they have been quoted for easy reference.

Theorem 2.1 [Mullineux [6]]
Suppose $n \geq 9$ and A_n lies in the finite group G with the 3-Cycle Property. Then G possesses a normal elementary Abelian 2-subgroup X such that $X \bigcap A_n = 1$ and G is a semi-direct product XA_n.

Theorem 2.2 [Feit-Thompson [3]]
Let G be a finite non Abelian group with a self-centralizing subgroup of order 3. Then G is isomorphic to one of the following:

(i) MD_6, a semi-direct product, where M is a nilpotent group, D_6 is a dihedral group of order 6;

(ii) YA_5, a semi-direct product, where Y is an elementary Abelian 2-group;

(iii) $PSL(2,7)$.

We have the following corollary to the Feit-Thompson theorem given above:
Corollary 2.2 [Feit-Thompson [3]]
If G is a finite simple group with a self-centralizing element of order 3, then G is isomorphic to A_5 or $PSL(2,7)$.

Theorem 2.3 [Bryson [2]]
Let G be a finite group with an element of order 3 whose centralizer is of order 9. Suppose also that G has an Abelian Sylow 3 -subgroup of order 9 and has two classes of elements of order 3. Then G is isomorphic to either A_6 or A_7.

Theorem 2.4 [Higman [5]]
Let the finite group G have a normal 2-subgroup Q such that G/Q is a dihedral group of order 6. Let an element s of G of order 3 act fixed-point-freely on Q, and let P be a Sylow 2-subgroup of G. Then:
(i) Q is of class at most 2;

(ii) if A is an Abelian subgroup of Q, then $\langle A, A^s \rangle$ is also Abelian;
(iii) if $|Q| > 4$, the class of Q is less than the class of any other subgroup of P of index 2^n.

Theorem 2.5 [Stewart [7]]
Let G be a finite group with a normal 2-subgroup Q such that G/Q is isomorphic to $PSL(2,2^n)$, $n \geq 2$, and suppose an element of G of order 3 acts fixed-point-freely on Q. Then:
(i) Q is an elementary Abelian and is the direct product of minimal normal subgroups of G each of order 2^{2n};
(ii) The Sylow 2-subgroup P of G is of class 2, and if $|Q| > 2^{2n}$, Q is the only Abelian subgroup of P of index 2^n.

Proposition 2.6 [Proposition 4.2 in [7]]
Let G be a finite group, H a normal subgroup of G with G/H isomorphic to $PSL(2,2^n)$, $n \geq 2$. Suppose an element t of G of order 3 acts fixed-point-freely on H. Then H is an elementary Abelian 2-group.

3 Finite Groups Having p-Cycle Property

We consider a non-Abelian finite group G with a p-cycle property, where p is prime, defined as follows: G is a finite group containing a non-Abelian simple group H where H has an element x of prime order p such that $C_G(x) \leq H$. We first define the following:
Definition 2
The Fitting subgroup of G, denoted by $F(G)$, is the subgroup generated by all nilpotent normal subgroups of G
Definition 3
A group H is subnormal in G if there exists a series
$H = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots G_n = G$.
Definition 4
A group X is quasisimple if $X = X'$ and $X/Z(X)$ is simple. In other words, X is quasisimple if X is perfect and $X/Z(X)$ is simple.
Definition 5
The components of a group X are its subnormal quasisimple subgroups.
We write $Comp(X)$ for the set of all components of X and $E(X) = \langle Comp(X) \rangle$, the subgroup generated by the set of components of X.
Definition 6
The generalized Fitting subgroup, denoted by $F^*(G)$, of G is the characteristic subgroup of G defined by $F^*(G) = F(G)E(G)$.

In [1], Theorem 31.12, it is proved that $F^*(G)$ is the central product of $F(G)$ with $E(G)$. We now prove the proposition 1:

Proposition 1
Let G be a finite group containing a subgroup H such that H is a non-Abelian simple group, and H has an element x of prime order p with $C_G(x) \leq H$. Then $O_{p'}(G) = F(G)$ and $G/O_{p'}(G)$ is a non-Abelian simple group.

Proof
Certainly, $H \bigcap F(G) = 1 = C_{F(G)}(x)$, and so $F(G) \leq O_{p'}(G)$. So $H \bigcap O_{p'}(G) = 1 = C_{Op'(G)}(x)$. By Thompson [4],Theorem 2.1, page 337, $O_{p'}(G)$ is nilpotent, and so $O_{p'}(G) = F(G)$. Let $\overline{G} = G/O_{p'}(G)$. Then $\overline{H} \cong H$, and $C_{\overline{G}}(x) = \overline{C_G(x)}$, so $\overline{G}$ satisfies the conditions of the proposition. We may assume that $O_{p'}(G) = F(G) = 1$ from now on. In this case, $1 \neq F^*(G) = E(G)$ is a direct product of non-Abelian simple groups. By Thompson [4], Theorem 2.1, $C_{F^*(G)}(x) \neq 1 \neq H \bigcap F^*(G)$, and so $H \leq F^*(G)$. Let $F^*(G) = K_1 \times K_2 \times, ..., \times K_r$, where $K_1, K_2, ..., K_r$ are non-Abelian simple groups. For each i, K_i is normalised by H, and therefore $C_{K_i}(x) \neq 1$. Accordingly, $H \bigcap K_i \neq 1$ which forces $H \leq K_1$, and so $r = 1$.

In the subsequent sections, we establish the results for particular cases of the p-cycle property, where H is isomorphic to A_n, for $n = 5$ or $n = 6$ and $p = 3$ if G is simple. Then corollaries that generalize the results when G is not simple are proved for each case.

4 The Case $n = 5$

We now prove:
Theorem 4.1
Let G be a finite simple group containing a subgroup $H \cong A_5$ such that $C_G(x) \leq H$, where $x \in H$ is the pre-image of $(123) \in A_5$ in the isomorphism. Then $G \cong A_5$.
Proof
From the statement of the theorem, we deduce that $C_G(x) = C_H(x) \cong \langle x \rangle$ as (123) is a self-centralizing cycle in A_5. It follows that G has a self-centralizing cyclic subgroup of order 3. Applying Theorem 2.2 and Corollary 2.2, G must be isomorphic to one of the following:
(i) A_5,
(ii) $PSL(2,7)$.

Since G contains $H \simeq A_5$, G can not be isomorphic to $PSL(2,7)$ because $\mid PSL(2,7) \mid = (7^2-1)7 = 48.7 = 336$ and $\mid A_5 \mid = \frac{5!}{2} = 5.4.3 = 60$. So $|A_5| \nmid |PSL(2,7)|$. Thus case (ii) can not hold. Only case (i) holds. Hence G is isomorphic to A_5, completing the proof.

Corollary 4.1
Let G be a finite group containing a subgroup $H \cong A_5$ such that $C_G(x) \leq H$, where x is the pre-image of (123) in A_5 in the isomorphism. Then $G \cong Q.K$, an extension of an elementary Abelian 2-group Q by a group $K \cong A_5$.

Proof
Let $\overline{G} = G/O_{3'}(G)$; and $H \leq G$ with $H \cong A_5$. Then $\overline{H} \leq \overline{G}$, where $\overline{H}$ is the image of H in $G/O_{3'}(G)$. By Proposition 1,
$O_{3'}(G) = F(G)$, and $G/O_{3'}(G)$ is non Abelian simple group. $\overline{H} \leq \overline{G}$ and $C_{\overline{G}}(\bar{x}) \leq \overline{H}$, where $\bar{x}$ is the image of x in $G/O_{3'}(G)$. By Theorem 4.1, we have $\overline{G} \cong A_5$. That is, $G/F(G) \cong A_5$. Since $F(G)$ is nilpotent, it is a direct product of its Sylow subgroups. Any element of order 3 in G lies in a subgroup isomorphic to A_5 and A_5 acts on the group $F(G)$ with the element of order 3 acting fixed-point-freely. By Stewart's result, Proposition 2.6, as $A_5 \simeq PSL(2,4)$, $F(G)$ is an elementary Abelian 2-subgroup. Then $G/Q \cong A_5$, where $Q = F(G)$ is an elementary Abelian 2-subgroup, and hence we have $G \cong Q.K$, an extension of an elementary Abelian 2-group Q by a group $K \cong A_5$.

5 The Case $n = 6$

We establish the following result:
Theorem 5.1 Let G be a finite simple group containing $H \cong A_6$ such that for $x \in H$ where x is the pre-image of (123) such that G satisfies the 3-cycle property. Then G is isomorphic to either A_6 or A_7.
Proof
Let $P =< x, y >$ where x is the pre-image of (123) and y is the pre-image of (456). Then P is an elementary Abelian 3-subgroup of G. If $x \in L$, where $L \leq H$, and $g \in N_G(L)$, then $x^g \in H$ by the 3-cycle property so that x and x^g must be conjugate in H. This implies $x^g = x^h$ for some $h \in H$ so that $x^{gh^{-1}} = x$. This implies $gh^{-1} \in H$ by the 3-cycle property. Since $h \in H$, $g \in H$ also so that $N_G(L) < H$. But in H, $N_G(P) = N_H(P) =< x, y >= P$ so that P is a Sylow 3-subgroup of its own normalizer, and hence it is a Sylow 3-subgroup of G. Then $C_G(P) = P$ so P is a self-centralizing elementary Abelian Sylow 3-subgroup of G of order

9. Applying Theorem 2.3, G is isomorphic to A_6 or A_7.

Corollary 5.1
Let G be a finite group containing a subgroup $H \cong A_6$ such that G satisfies the 3-cycle property. Then $G \cong QF$, an extension of an elementary Abelian 2-group Q by a group F isomorphic to A_6 or A_7.

Proof
We apply a similar argument and results as in the proof of Corollary 4.1, with $\overline{G} = G/O_{3'}(G)$; and $H \leq G$ with $H \simeq A_6$. That is, we let $\overline{G} = G/O_{3'}(G)$; and $H \leq G$ with $H \simeq A_6$. Then $\overline{H} \leq \overline{G}$, where $\overline{H}$ is the image of H in $G/O_{3'}(G)$. By Proposition 1, $O_{3'}(G) = F(G)$ and $\overline{G}$ is a non-Abelian simple group. We have $\overline{H} \leq \overline{G}$, $C_G(\overline{x}) \leq \overline{H}$ and $\overline{x}$ is not conjugate to $\overline{xy}$ in $\overline{G}$. By Bryson's results, Theorem 2.3, we have $\overline{G} \cong A_6$ or $\overline{G} \simeq A_7$. That is, $G/F(G) \cong A_6$ or $G/F(G) \simeq A_7$. Any element of order 3 in A_6 lies in a subgroup isomorphic to A_5, and A_5 acts on the group $F(G)$ with the element of order 3 acting fixed-point-freely. By Stewart's result, Proposition 2.6, $F(G)$ is an elementary Abelian 2-group Q. Whence, $G/Q \cong A_6$ or $G/Q \cong A_7$. This implies either $G \cong QA_6$ or $G \cong QA_7$, an extension of an elementary Abelian 2-group Q by either A_6 or A_7. This concludes the proof of the main result.

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