A Classification of Fuzzy Subgroups of Finite Abelian Groups

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Abstract

The knowledge of fuzzy sets and systems has become a considerable aspect to apply in various mathematical systems. In this paper, we apply a knowledge of fuzzy sets to group structures. We consider a fuzzy subgroups of finite abelian groups, denoted by G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, where $\mathbb{Z}$ is an integer, p and q are distinct primes and m, n are natural numbers. The fuzzy subgroups are classified using the notion of equivalence classes. In essence the equivalence relations of fuzzy subsets X is extended to equivalence relations of fuzzy subgroups of a group G. We then use the notion of flags and keychains as tools to enumerate fuzzy subgroups of G. In this way, we characterized the properties of the fuzzy subgroups of G. Finally, we use maximal chains to construct a fuzzy subgroups-lattice diagram for these groups of G.

Keywords: Finite Abelian Group, Fuzzy Subgroup, Equivalence Relation, Fuzzy Set, Flag and Keychain.

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1 Introduction

Since the notion of fuzzy sets was introduced by Zadeh in 1965 (see [24]), there have been attempts to extend useful mathematical notions to this wider setting replacing sets by fuzzy sets (see also [8], [12] and [13]). Another important development was the study of finite abelian groups in the field of Algebra with relation to fuzzy sets. A major significance in dealing with fuzzy sets can be achieved through a principle of working with fuzzy (real) numbers. The basic arithmetic operations involving fuzzy number allow us to exercise the operations between them. Many laws that hold for the arithmetic of real numbers also hold for the fuzzy intervals, but the distribution property holds only when restricted. We assume that if we use the extension principle then we can apply many operations to this new mathematical system. In this regard, there are many forms of fuzzy numbers that are associated with the following formats: *sine numbers, bell shape, polygonal, trapezoids, and triangular*.

The paper focus on a system that classify the fuzzy subgroups of a finite abelian group G. This is an extension that relates two things with common features. In some of the exsting literatures (see for instance [23] and [22]) isomorphism was appropriate to use for classification. However, in the paper [13], Murali and Makamba disputed that fact and proposed that equivalence relation is much appropriate. Equivalence relations provide a conducive setting for classifying fuzzy subgroups of a group G. In his paper, Denga [5], extend the discussion under which equivalence relation of fuzzy sets is quantified by their level sets. Therefore, the use of equivalence relation facilitate fuzzy subgroups of finite abelian groups to be classified in some special cases. In this paper, we contribute to the debate by applying this notion to an abelian group G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ which is an extention of some ground work pioneered by Murali, Makamba and Ngcibi (see for intance [13], [14], [12] and [19]). In their paper [15], they considered a finite abelian group of the form G = $\mathbb{Z}_p$, where p is a prime. In this case they classify fuzzy subgroups of this p - group. Furthermore, they considered a group of the form G = $\mathbb{Z}_{p^n} + \mathbb{Z}_q$, where p and q are distinct primes and n is any natural number. In another direction Ngcibi [19] conducted research on fuzzy subgroups of G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$.

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These information serves as the fundamental basis for the equivalence of fuzzy subgroups of a finite abelian group G. Our discussion extend this by considering two distinct primes p and q, whose power is n and m, respectively. The discussion on equivalence relations of fuzzy subgroups of a group G and its equivalence relations of fuzzy subgroups of a group G is strengthen with the notion of flags and keychains projected as tools for enumerating fuzzy subgroups of G. The flags and keychains are essential for the characterization of the properties of the fuzzy subsets of a set X. These keychains represent the equivalence classes of the fuzzy subsets (see Murali and Makamba [16]). Then, we use maximal chain to construct a fuzzy subgroup-lattice diagrams for the groups of G. This is essential for ordering items with a desire to achieve ordered patterns. Hence subgroups on the lattice is arranged according to the number of elements they occupy (see for more information [8]). The main aim with lattice - diagram is to ascertain the nature of meet and join of fuzzy subgroups. Finally, we give some of the valuable contributions in the literature of classifiying fuzzy sets, are the notable work in [5], [4] and [20].

1.1 Main Questions and Article Outline

In this subsection, we pose the main questions and provide an outline of the article.

1.1.1 Main Questions

In this article on classification of fuzzy subgroups of finite abelian group, we answer the following questions.

Question 1.1 *Can we classify the fuzzy subgroups of a finite abelian group using equivalence relations?*

Question 1.2 *Can we use flags and keychains to characterize the fuzzy subgroups of a group G?*

1.1.2 Article Outline

The rest of the paper is organized as follows. Section 1 provides introductory remarks and also related literature. The notion of fuzzy sets and systems is given in Section 2.1. Issues pertaining to finite abelian groups and their properties are discussed in Section 2.2. Section 3 provides the fundamental part of the discussion with equivalence relations. These equivalence relations are used to classify the fuzzy subgroups of a group G. The classification of fuzzy subgroups of G is discussed in Section 4. Finally, we provide the discussion, concluding remarks and future directions for the paper in Sections 5 and 6, respectively.

2 Preliminaries

2.1 Fuzzy Sets and Systems

2.1.1 Fuzzy Sets

In this Subsubsection, we discuss fuzzy sets and operations with them. In order to differentiate between crisp sets and fuzzy sets, we consider M = $\{a_i|1 \le i \le 10\}$ where a_i are members of a class X of object. We represent M with characteristic function, i.e. $\forall x \in X$

$$\Delta_M(x) = \begin{cases} 1, & if \quad x \in M, \\ 0, & otherwise. \end{cases} \tag{2.1}$$

Definition 2.1 *A fuzzy set of X is a pair (X,μ) where I = [0,1]. We shall denote by I^X the class of fuzzy subsets of X.*

Suppose that $x \in X$, then $0 \le \mu_A(x) \le 1$ is called *the degree of membership of X* to the fuzzy subset A of X. If $\mu_A(x)$ = 1, then x belongs to A absolutely. In case $\mu_A(x)$ = 0, then x does not belong to A absolutely. However, if $\mu_A(x)$ takes only $\{0,1\}$, $\forall x \in X$, then A is called a *crisp set*.

Definition 2.2 *A partition is a kernel of fuzzy sets if that partition of X whose blocks consists of elements with the same membership is a collection of elements in the Universe discourse of X, of equal membership values.*

Definition 2.3 *A fuzzy set A of X, whose membership function is defined by*

$$\mu_A(x) = \begin{cases} \lambda, & if \quad x = a, \\ 0, & otherwise, \end{cases} \tag{2.2}$$

where $0 < \lambda < 1$*, is called a* fuzzy point *of X.*

If $\lambda = 1$, then one would have a crisp set. We denote a fuzzy point by a^{λ}. Also an empty fuzzy set is the one whose membership value is 0, for all $x \in X$.

2.1.2 Alpha - cuts

In this Subsubsection, we discuss about weak alpha - cuts and strong alpha - cuts. Consider A to be a fuzzy subset of X, and $\alpha \in [0,1]$. Then, we define a weak α - cut as

$$A^{\geq\alpha} = \{x \in X | \mu_A(x) \geq \alpha\}. \tag{2.3}$$

Similarly, we define a strong α - cut as

$$A^{>\alpha} = \{x \in X | \mu_A(x) > \alpha\}. \tag{2.4}$$

2.1.3 Operations with fuzzy sets

In this Subsubsection, we discuss about fuzzy sets operations and relate them to the notion of α - cuts.
Let A and B be two fuzzy sets, then we define the union of two fuzzy sets as

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x). \tag{2.5}$$

The intersection of two fuzzy sets A, B and the complement of the fuzzy set A, denoted by A^c (relative to X) are defined by

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x), \forall x \in X, \tag{2.6}$$

$$\mu_A(x) = 1 - \mu_A(x), \forall x \in X, respectively. \tag{2.7}$$

We relate the above operations with the weak α - cut as below, however the relation with strong α - cut is worked out in a similar fashion.

$$(A \cap B)^{\geq\alpha} = A^{\geq\alpha} \cap B^{\geq\alpha}, \tag{2.8}$$

$$(A \cup B)^{\geq\alpha} = A^{\geq\alpha} \cup B^{\geq\alpha}, \tag{2.9}$$

$$(A^c)^{\geq\alpha} = A^{\leq 1-\alpha} = \{x \in X | \mu_A(x) \leq 1 - \alpha\}. \tag{2.10}$$

2.1.4 Inclusion for fuzzy sets

Take two fuzzy subsets A and B of X. We call A a subset of B, denoted by $A \subset B \Leftrightarrow$ the membership degree $\mu_A(x)$ is never greater than the membership degree $\mu_B(x)$, i.e.

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \quad \text{for} \quad \text{all} \quad x \in X. \tag{2.11}$$

2.1.5 Fuzzy numbers and their Intervals

In this Subsubsection, we introduce the definition and notations of fuzzy numbers and their intervals. Let $F(\mathbb{X})$ be the class of all fuzzy subsets of the Universe of discourse X and fuzzy (real) numbers, $\mathbb{F}(\mathbb{X})$ are imprecise values in the interval [0,1].

Definition 2.4 *A fuzzy set $A \in \mathbb{F}(\mathbb{X})$ is called a fuzzy (real) number, if and only if A is* convex *and if there exists exactly one real number a with $\mu_A(a)$ = 1.*

The shape of membership functions of fuzzy numbers are convex and we define a convex relation to fuzzy numbers as follows.

Definition 2.5 *A fuzzy number $A \in \mathbb{F}(\mathbb{X})$ is called* convex*, if and only if all (strong) α - cuts of A are intervals, i.e. themselves convex sets in the usual sense.*

In case of " *betweenness*", we have a notion of interval $[a,b]$ with end points a,b, so that if $c \in [a,b]$ then $\mu_A(c) \geq \min\{\mu_A(a), \mu_A(b)\}$. Therefore, when A is only convex and normal then A is called a *fuzzy interval*.

2.2 Finite abelian groups

This Section provides a brief background about finite abelian groups, definitions as well as the theory necessary for what is to follow.

2.2.1 Cyclical groups of any finite order

Let G be a group and $a \in$ G . The set $< a > = \{a \in G | n \in \mathbb{Z}\}$ is called *cyclic subgroup* generated by a. The group G is called a *cyclic group* if there exist an element $a \in G$ such that G = $< a >$. In this case a is called a generator of G. A group G is abelian if $ab = ba$ for all $a,b \in G$. Cyclic groups are abelian, but the converse is not true. The following proof shows that all cyclic groups are abelian:

Let G be a cyclic group and a be a generator of G. Take $c,d \in G$. Then there exist $x,y \in \mathbb{Z}$ such that $c = a^x$ and $d = a^y$. Since $cd = a^x a^y = a^{x+y} = a^y a^x = dc$, then, G is abelian.

Definition 2.6 *The order of a finite group is the number of its elements, while the order of an element a ∈G is the smallest positive integer n such that a^n = e.*

If the subgroup H = $< a >$ from the definition is finite, and $a^n = a^m$ for $n,m \in \mathbb{N}$ and $n > m$, then $a^n.a^{-m} = a^m.a^{-m}$. Therefore $a^{n-m} = e$. Furthermore, if $n > m$ and the subgroup $< a >$ is finite, the set of integers $\mathbb{Z} = \{n \in \mathbb{N} : a^n = e\}$ is non empty, by Well Order Principle, there is at least a positive integer r such that $a^r = e$. Therefore $o(a) = r$ such that $< a > = \{a^0 = e, a, a^2, \cdots, a^{r-1}\}$.

Proposition 2.7 *Let G be a finite cyclic group with 0(G) = n be a finite number. If $d|n$, then there exists exactly one subgroup of G of order d.*

Subgroups of a cyclic group are cyclic and all groups of prime order are cyclic. A simple group is a group whose only normal subgroups are the trivial group of order 1 and an improper subgroup consisting of the entire original group.

Definition 2.8 *The following argument shows that all cyclic groups are* simple *if and only if the number of its elements is a prime. The simplest abelain groups are the cyclic groups of order n = 1 or of a prime order p.*

2.2.2 Isomorphism

Isomorphism allows us to treat certain groups as being alike.

Definition 2.9 *Let G be a group with operation $\star$ and let H be a group with operation $\bullet$. An isomorphism of G onto H is a mapping $\theta : G \to H$ that is one - to - one, onto and also satisfies*

$$\theta(a \star b) = \theta(a) \bullet \theta(b) \tag{2.12}$$

for all $a, b \in G$.

The following theorem shows that any group isomorphic to an abelian group must also be abelian.

Theorem 2.10 *If G and H are isomorphic groups and G is abelian, then H is abelian.*

Proof.

We define the operation on G to be $\star$ and on H to be $\bullet$, respectively. And we set $\theta : G \to H$ to be an isomorphism. Then for all $x, y \in H$, there are elements $a, b \in G$ such that $\theta(a) = x$ and $\theta(b) = y$.

Since θ preserves the operation and G is abelian,

$$x \bullet y = \theta(a) \bullet \theta(b) = \theta(a \star b) = \theta(b \star a) = \theta(b) \bullet \theta(a) = y \bullet x. \tag{2.13}$$

This shows that H is abelian. □

2.2.3 Equivalence relation on groups

In this study we discuss a connection between groups.

The binary relation $\sim$ on A is said to be an equivalence relation on A if for all $a, b, c,$ in A

1. a $\sim a$
2. $a \sim b \to b \sim a$
3. $a \sim b$ and $b \sim c \to a \sim c$

The first of these three properties is called *reflexivity*, the second *symmetry* and the third *transitivity*.

When working with an equivalence relation on a set A, it is often useful to have a complete set of equivalence class representatives.

2.2.4 Equivalence class

Definition 2.11 *If A is a set and $\sim$ is an equivalence relation on A, then the equivalence class of $a \in A$ is a set $\{x \in A | a \sim x\}$.*

We denote the equivalence class containing a by $[a]$.

2.2.5 Cyclic groups and its rank

Definition 2.12 *A* rank *is defined as the cardinality of the largest set of linearly independent elements of the group.*

The integer and rational numbers have rank one, as well as every subgroup of the rationals.

The proposition below give rise to a condition that a finite abelian group can be expressed as a direct product of its Sylow p - subgroups.

Proposition 2.13 *Let G be a finite abelian group. Then G is isomorphic to a direct product of cyclic groups*

$$\mathbb{Z}_{n1} \times \mathbb{Z}_{n2} \times \cdots \times \mathbb{Z}_{nk},$$

such that $n_i | n_{i-1}$ for i = 2, 3, $\cdots$,k.

2.2.6 Fundamental theorem of finite abelian groups

For each non zero finite abelian group G, there is exactly one list $m_1, m_2, \cdots, m_k$ of integers $m_i > 1$, each a multiple of the next, for which there is an isomorhism

$$G = \mathbb{Z}_{m1} \oplus \cdots \oplus \mathbb{Z}_{mk}.$$

In this description, the first integer m_1 is the least positive integer $m = m_1$ with $m\mathbb{G} = 0$, while the product $m_1 m_2 \cdots m_k$ is the order of G.

2.2.7 Lattice

In this Subsection, we discuss about lattice of subgroups.

2.2.8 Lattice of Subgroups

Definition 2.14 *$(S, \leq)$ is a lattice if and only if $(S, \leq)$ is a partially ordered set and each pair of elements $x, y \in S$, $\{x, y\}$ has a least upper bound and a greatest lower bound.*

The book by P. Crawley and R.P. Dilworth on the algebraic theory of lattices defines a sublattice, which is a non empty subset M of a lattice L which is a sublattice of L if $x \vee y$, $x \wedge y \in$ M, and $x, y \in$ M where $\wedge$ is the meet and $\vee$ is the joint. If we speak of a homomorphism, which is a mapping f of lattice L to a lattice M, then for all $x, y \in L$, $f(x \vee y)$ = $f(x) \vee f(y)$ and $f(x \wedge y)$ = $f(x) \wedge f(y)$.

2.2.9 Complete Lattice

Definition 2.15 *A complete lattice L is a partially ordered set (poset) in which every subset has a least upper bound and a greatest lower bound.*

Consequently, a complete lattice contains top element $(1 = \vee L)$ and bottom elements $(0 = \wedge L)$.
The following result describes some properties of subgroup lattices in general. Let G be a group. Then $(L(G), \subseteq)$ is a lattice.
The theorem below gives a subgroup-lattice characterization of groups.

Theorem 2.16 *If G is a group such that*

$$L(G) \cong L(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}), \quad \text{then} \quad G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}.$$

3 Equivalence Relations

In this Section we discuss equivalence relations on fuzzy subsets and fuzzy groups. The equivalence relations provide a setting for classifying the fuzzy subgroups of G.

3.1 Equivalence relation on fuzzy subsets

In this Subsection, we provide a definition for equivalence relation and also their properties.

Definition 3.1 *We define an equivalence relation $\sim$ on I^X, where I = [0,1] and X is a set defined by*

$$\mu \sim \nu \quad \text{if} \quad \text{and} \quad \text{only} \quad \text{if} \tag{3.14}$$

1. for all $x, y \in X$, $\mu(x) > \mu(y) \Leftrightarrow$ if $\nu(x) > \nu(y)$.
2. $\mu(x) = 0 \Leftrightarrow \nu(x) = 0$.
3. $\mu(x) = 1 \Leftrightarrow \nu(x) = 1$.

Part 1 in defintion (3.1) is straight forward. However, Part 2 is important for the equivalence relation application and we justify it in the example below.

Example 3.2 *Consider $X_3 = \{\omega, s, s^2, t, st, s^2t\}$, and define two fuzzy subsets β and $\overline{\beta}$.*

$$\beta(x) = \begin{cases} 1, & if \quad x = \omega, \\ \frac{1}{2}, & if \quad x = s, s^2, \\ \frac{1}{3}, & otherwise. \end{cases} \tag{3.15}$$

$$\overline{\beta}(x) = \begin{cases} 1, & if \quad x = \omega, \\ \frac{1}{2}, & if \quad x = s, s^2, \\ 0, & otherwise. \end{cases} \tag{3.16}$$

Then Supp $\beta \neq$ Supp $\overline{\beta}$, i.e. $\frac{1}{3} \neq 0$, meaning that condition 2 is not satisfied, while the rest of the conditions are satisfied for all $x \in X$. Thus $\beta \sim \overline{\beta}$.
In this paper, an equivalence class containing μ is denoted by $[\mu]$, with respect to the relation above. The following proposition in [20] establishes that $\sim$ is an equivalence relation on $\tau_L(x)$, the collection of all fuzzy subsets on X whith co-domain is L, a lattice. L is a partially ordered set with unique least upper bounds and greatest lower bounds.

Proposition 3.3 *(Seselja and Tepavcevic, [20])*
The relation $\sim$ is an equivalence relation on $\tau_L(x)$.

In the example below we illustrate that the converse of the proposition is not true.

Example 3.4 *Let $Z_3 = \{\omega, s, s^2, t, st, s^2t\}$, and the fuzzy subsets β and $\overline{\beta}$ are defined as:*

$$\beta(y) = \begin{cases} 1, & if \quad y = \omega, \\ \frac{1}{2}, & if \quad y = s, \\ \frac{1}{3}, & otherwise. \end{cases} \tag{3.17}$$

$$\overline{\beta}(y) = \begin{cases} 1, & if \quad y = \omega, \\ \frac{1}{2}, & if \quad y = s, t, \\ 0, & otherwise. \end{cases} \tag{3.18}$$

From the above you can see that the images and support of two fuzzy subsets are equal, so $\beta(s) > \beta(st)$, then $\overline{\beta}(st) \not> \beta(st) \Rightarrow \beta \nsim \overline{\beta}$.
The proposition below highlights the fact that if the α - cuts of two distinct fuzzy subsets are equal, then the relation defined in definition 3.1 holds between those two fuzzy subsets.

Proposition 3.5 *(Murali and Makamba, [13])*
Let β and $\overline{\beta}$ be two fuzzy subsets of X. Suppose that for each $l > 0$ there exists $k > 0$ such that $\beta^l = \overline{\beta}^k$. Then $\beta \sim \overline{\beta}$.

3.2 Flags and keychains

In this Subsection we provide definitions for flags and keychains. Flags and keychains are used to characterize the properties of fuzzy subsets of X. In this subsection we fix n.

3.3 Flags

Definition 3.6 *A flag ♠ on a set X is a maximal chain of subsets of X such that $X_0 \subset X_1 \subset \cdots \subset X_n = X$.*

The maximal chains are given by permutations of the elements of X. The subsets of X are permuted into n and the length of the maximal chain of X is $(n+1)$.

3.4 Keychains

Definition 3.7 *An n - chain is called a* keychain *if* $1 = \overline{\beta}_0 \geq \overline{\beta}_1 \geq \overline{\beta}_2 \geq \cdots \geq \overline{\beta}_n \geq 0$. *We denote a keychain by ♣.*

The $\overline{\beta}_i^{\prime}s$ are called *pins*. The pins that are interlocked are called *components*. In our case " 1" is not a component and a keychain with k - distinct components is called a *k-pad*, where $(1 \leq k \leq n)$. The $\overline{\beta}_i^{\prime}s$ are classfied either by " = " or " $>$ ".

Example 3.8 *Consider*
1. $1 > \overline{\beta}_1 = \overline{\beta}_2 > \overline{\beta}_3 = \overline{\beta}_4 = \overline{\beta}_5 > \overline{\beta}_6 > 0.$
We have 4 - pads keychain of a 7 - chain.
2. $1 = \overline{\beta}_1 = \overline{\beta}_2 = \overline{\beta}_3 > \overline{\beta}_4 > \overline{\beta}_5 = \overline{\beta}_6 = \overline{\beta}_7.$
We have 3 - pad keychain of a 7 - chain.

Definition 3.9 *A paddity of a component is the number of pins found in the interlocked position forming the component.*

In Example 3.8(1) above, we see that the paddities of the components are 2,3, and 1.

Definition 3.10 *The index of a k - pad keychain is the set of paddities of various components of the keychain in which singleton components are ignored for the sake of simplicity.*

From Example 3.8(1) above, the partition of 7 given by 2+3+1+1 corresponds to a 4 - pad keychain whose index is (2,3).

Definition 3.11 *A pinned - flag is a pair consisting of a flag ♠ and a keychain ♣.*

The α - cuts $(0 \leq \alpha \leq 1)$ of fuzzy subsets belonging to the same equivalence class can be represented as pinned - flag. We construct a fuzzy subset μ on X corresponding to a pinned - flag on X as

$$0^1 \subset X_1^{\overline{\beta}_1} \subset \cdots \subset X_n^{\overline{\beta}_n} \tag{3.19}$$

in the following manner:

$$\mu(y) = \begin{cases} 1, & if \quad x = 0, \\ 0, & if \quad x \in X_1 \setminus \{0\}, \\ \vdots & \\ \overline{\beta}_n & if x \in X_n \setminus X_{n-1}. \end{cases} \tag{3.20}$$

In the following proposition Murali and Makamba ([13]) discuss conditions of pinned - flags corresponding to two fuzzy subsets with equivalent relations.

Proposition 3.12 *(Murali and Makamba, [13])*
Suppose the pinned - flags corresponding to two fuzzy subsets β and $\overline{\beta}$, are

$$(\spadesuit_{\beta}, \clubsuit_{\beta}) : P_0^1 \subset P_1^{\lambda_1} \subset \cdots \subset P_n^{\lambda_n}, \tag{3.21}$$

and

$$(\spadesuit_{\overline{\beta}}, \clubsuit_{\overline{\beta}}) : Z_0^1 \subset Z_1^{\sigma_1} \subset \cdots \subset Z_m^{\sigma_m}, \tag{3.22}$$

where the λ_i and the σ_i are all distinct. Then $\beta \sim \overline{\beta}$ on $X \Leftrightarrow$:
(i) n = m,
(ii) P_i = Z_i for i = 0, 1, $\cdots$,n,
(iii) λ_i = $\lambda_j \Leftrightarrow \sigma_i = \sigma_j$ for all $1 \leq i, j \leq n$ and $\lambda_k = 0 \Leftrightarrow \sigma_k = 0$ for some $1 \leq k \leq n$.

The proposition clarifies that the two pin flags are related.

3.5 Equivalence of fuzzy points

In this Subsection, we state the relationship between equivalent fuzzy subsets and equivalent fuzzy points. We look at the results given by distinct keychains relative to equivalence of fuzzy points. In this case, we take a chain of fuzzy subsets by specifying a crisp point from the chain and obtain distinct keychains that are being hosted by the crisp point.

Example 3.13 *Let $g \in \mathbb{Z}$ such that*

$$\mathbb{Z}_0 \subset \{y\} \subset \mathbb{Z}. \tag{3.23}$$

From the above we can deduce about seven different keychains and they represent the equivalence classes of fuzzy subsets. In the example below we discuss the nature of a fuzzy point belonging to a fuzzy subset under the defined equivalence.

$$\mu(y) = \begin{cases} 1, & if \quad y = y, \\ \frac{1}{2}, & if \quad y = y_2, \\ \frac{1}{3}, & otherwise. \end{cases} \tag{3.24}$$

$$\nu(y) = \begin{cases} 1, & if \quad y = y, \\ \frac{1}{4}, & if \quad y = y_2, \\ \frac{1}{5}, & otherwise. \end{cases} \tag{3.25}$$

We can deduce that $\mu(y) > \mu(x) \Leftrightarrow \nu(y) > \nu(x)$, for $x, y \in X$ and Supp μ = Supp ν. So, $\mu \sim \nu$, but the fuzzy point y_2^{λ} for $\frac{1}{2} > \lambda > \frac{1}{3}$ belongs to μ and fails to belong to ν.
The notion of fuzzy points and fuzzy points belonging to fuzzy subsets did prove very useful to the study of equivalence of fuzzy subsets. In light with this Murali and Makamba, [13], came up with a definition of a fuzzy point belonging to an equivalent class of fuzzy subsets in terms of pinned-flags which is more useful to the study of equivalence of fuzzy subsets of X.

Definition 3.14 *Suppose,* $(\spadesuit_\mu, \clubsuit_\mu)$ *is the pinned - flag corresponding to an equivalence class of fuzzy subset* $[\mu]$ *given by*

$$(\spadesuit_\mu, \clubsuit_\mu) : X_0^1 \subset X_1^{\lambda_1} \subset \cdots \subset X_n^{\lambda_n}. \tag{3.26}$$

Then, we say an equivalence class of fuzzy point $[a^\lambda]$ *belongs to* $[\mu]$ *if and only if* $0 < \lambda \leq \lambda_i < 1$, *where* $0 < i \leq n$ *is the least index with the property* $a \in X_i$, *but* $a \notin X_{i-1}$.

The framework on equivalence of fuzzy subsets of X creates enabling conditions which allows us to study the equivalence of fuzzy subgroups of G.

3.6 Equivalence of fuzzy subgroups of G

In this subsection e is the identity of the group G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, where p, q are primes and n, m are natural numbers. The main focus of this subsection is to study the fuzzy subgroups of finite abelian groups.

Definition 3.15 *Two fuzzy subgroups* μ *and* ν *of G are equivalent and we write* $\mu \sim \nu \Longleftrightarrow$
(i) for all $x, y \in G$, $\mu(x) > \mu(y) \Longleftrightarrow \nu(x) > \nu(y)$ *and*
(ii) $\mu(x) = 0 \Longleftrightarrow \nu(x) = 0$.

The proposition below set a framework for characterizing the maximal chains of the fuzzy subgroups of a group G.

Proposition 3.16 *If, in a group G, the length of the longest maximal subgroups chain with end-points* $\{e\}$ *and G is n, then the order of any fuzzy subgroup of G cannot be greater than* $n+1$.

With the aid of the above proposition, we consider the structure of maximal chains of the fuzzy subgroups of G.

3.7 Maximal chains of subgroups of G

The linear ordering of the lattice of subgroups provide a way of characterizing maximal chains in G. Here we permute $n+m$ objects of which n of them identical objects say $p \quad p \quad p \quad \cdots \quad p$ and m of them are identical objects say $q \quad q \quad q \quad \cdots \quad q$.
Let us consider a general case with $p^n q^m$, we can list

$$o \quad \underbrace{p \quad p \quad p \cdots \quad p}_{n} \qquad \underbrace{q \quad q \quad p \quad \cdots \quad q}_{m}. \tag{3.27}$$

The above listing gives rise to the following proposition.

Proposition 3.17 *The list of orders of subgroups of G gives rise to the following maximal chain*

$$\{0\} \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \cdots \subset \mathbb{Z}_{p^n} \subset \mathbb{Z}_{p^n q} \subset \cdots \subset \mathbb{Z}_{p^n q^m} \cong G, \tag{3.28}$$

where p and q are distinct primes while, m and n are two fixed positive integers.

Proof.

Suppose there is a subgroup H of G whose order is d, such that $\mathbb{Z}_{p^i} \subset H \subset \mathbb{Z}_{p^{i+1}}$, *for some* $0 \leq i \leq n-1$. *Then by Langrage theorem,* $d|p^{i+1}$. *This means* $d = p^i$ *for some* $j < i+1$. *Also,* $\mathbb{Z}_{p^i} \subset H \Leftrightarrow p^i|d$. *This implies that* $i < j < i+1$. *This is a contradiction.*

Similarly, if we insert H anywhere else in the equation, then the claim given by equation (3.28), holds the same conclusion. So the list of orders of subgroups of G gives rise to the above mentioned maximal chain. □

In the even of permuting subgroups of G various maximal chains are attainable as

$$0 \quad p \quad p \quad p \quad \cdots \quad p \quad q \quad q \quad q \quad \cdots \quad q. \tag{3.29}$$

If we shift $p's$ and $q's$ around we obtain other permutations, i.e.

$$0 \quad p \quad q \quad \cdots \quad p \quad q \quad p \cdots q. \tag{3.30}$$

By symmetry we can swop $p's$ and $q's$ without affecting the number of maximal chains. Consider the following maximal chain

$$\{0\} \subset H_1 \subset H_2 \subset \cdots \subset H_{n+m} = G. \tag{3.31}$$

From the above we deduce the following subgroup orders

$$1|d_1|d_2|\cdots|d_{n+m} = p^n q^m. \tag{3.32}$$

Therefore each d_i must be of the form $p^i q^i$, where $i \leq i_1 \leq n$ and $0 \leq i_2 \leq m$. It is easy to see that $\{0\}$ cannot be permutated, however for G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, we can permute it by $\frac{(n+m)!}{n!m!}$. This implies that G has $\frac{(n+m)!}{n!m!}$ maximal chains and are all of the same length with $(n+m+1)$ components.

3.8 Maximal chains of subgroups of $\mathbf{Z}_{72}$

In this Subsection, we take a specific case of G = $\mathbb{Z}_{72}$ = $\mathbb{Z}_{2^3} + \mathbb{Z}_{3^2}$ with p = 2, q = 3 and n = 3, m = 2. Now $\mathbb{Z}_{72}$ and $\mathbb{Z}_{2^3} + \mathbb{Z}_{3^2}$ are isomorphic to each other. In general not all subgroups of G are comparable with respect to containment or inclusion. Therefore the subgroups of G do not form a chain but a lattice.

We represent the subgroup - lattice diagram of G = 72 which consists of different chains of fuzzy subgroup of G below. The group G = $\mathbb{Z}_{72}$ has the following maximal chains with the following group inclusion

$$\begin{gathered}
\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_9 \subset \mathbb{Z}_{18} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{18} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72} \\
\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{18} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}
\end{gathered}$$

From the subgroups - lattice diagram of G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, we calculate the number of distinct fuzzy subgroups of G, using the counting principle, for small values of n and m.

3.9 Pinned - flag of G

We classify the fuzzy subgroups of G using keychains and various keychains are dinstinct from each other if they are made up of different pins.

Definition 3.18 *A pinned-flag of G is a pair* $(\spadesuit, \clubsuit)$ *consisting of a flag of subgroups and keychains.*

Given a keychain $1 \quad \lambda_1 \quad \lambda_2 \cdots \lambda_n$, where the $\lambda_i's$ are not all distinct, we can construct a fuzzy subgroup μ on G corresponding to a pinned - flag on G given by

$$0^1 \subset G_2^{\lambda_1} \subset \cdots \subset G_n^{\lambda_n}, \tag{3.33}$$

as follws:

$$\mu(x) = \begin{cases} 1, & if \quad x = 0, \\ \lambda_1, & if \quad x \in \mathbb{Z}_1 \setminus \{0\}, \\ \lambda_2, & if \quad x \in \mathbb{Z}_2 \setminus \mathbb{Z}_1, \\ \lambda_3, & if \quad x \in \mathbb{Z}_3 \setminus \mathbb{Z}_2, \\ \lambda_4, & if \quad x \in \mathbb{Z}_4 \setminus \mathbb{Z}_3, \\ \lambda_5, & if \quad x \in \mathbb{Z}_5 \setminus \mathbb{Z}_4. \end{cases} \tag{3.34}$$

If we take λ, β and γ to be real numbers in the interval [0,1], such that $0 < \gamma < \beta < \lambda \leq 1$. With one of the maximal chains of subgroups of G = $\mathbb{Z}_{72}$ is

$$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72}. \tag{3.35}$$

Then a fuzzy subgroup is obtained with membership values given by

$$1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 > \lambda_4 > \lambda_5 \tag{3.36}$$

where

$$\lambda_1 = \lambda, \lambda_2 = \lambda_3 = \beta, \lambda_4 = \gamma \quad \text{and} \quad \lambda_5 = 0. \tag{3.37}$$

4 Fuzzy subgroups of G

In this section, we discuss fuzzy algebraic structure via elements.

Definition 4.1 *Suppose G is a group, and* $\mu : G \to I$ *is a fuzzy subset of a group G, then* μ *is said to be a fuzzy subgroup of G if and only if* $\mu(ab) \geq \mu(a) \wedge \mu(b)$ *and* $\mu(a^{-1}) \geq \mu(a)$ *for all* $a, b \in G$.

The following proposition detailed some important properties of fuzzy subgroups.

Proposition 4.2 *1.* $\mu(a) = \mu(a^{-1}) \quad \text{for} \quad \text{all} \quad a \in G.$

2. $\mu(e) \geq \mu(a) \quad \text{for} \quad \text{all} \quad a \in G \quad \text{and} \quad e \in G.$

3. The $\alpha-$ cut μ^{α} of any fuzzy subgroup μ of G is a crisp subgroup of G for all α such that $0 \leq \alpha \leq \mu(e) \leq 1$.

i. The proofs for 1. and 2. are straight forward.

ii. We have to prove that $\mu^{\alpha} = \{\mu \in G : \mu(g) \geq \alpha\}$ is a crisp subgroup of G.

(a) Fix $g_1, g_2 \in \mu^{\alpha}$. Then $\mu(g_1) \geq \alpha$ and $\mu(g_2) \geq \alpha$. Therefore $\mu(g_1g_2) \geq \mu(g_1) \wedge \mu(g_2) \geq \alpha \wedge \alpha = \alpha$ for $g_1, g_2 \in G$. $\Rightarrow g_1g_2 \in \mu^{\alpha}$.

(b) Furthermore for any $\mu(g) \geq \alpha$, $\mu(g^{-1}) = \mu(g) \geq \alpha$, $\Rightarrow$ "$g \in \mu^{\alpha} \Leftrightarrow g^{-1} \in \mu^{\alpha}$".
Therefore μ^{α} is a crisp subgroup of G.

We define the fuzzy abelian group as:

Definition 4.3 *Suppose G is a group, and $\mu : G \to I$ is a fuzzy subgroup. Then μ is said to be a fuzzy abelian if $\mu(ab) = \mu(ba)$ for all $a, b \in G$.*

We only know that $\mu(ab) \geq \mu(a) \wedge \mu(b)$ and $\mu(ba) \geq \mu(b) \wedge \mu(a)$, but we cannot verify that these two are equal in general. Therefore the converse do not necessarily need to be true in general. Also $\mu(a^2) \geq \mu(a \cdot a) \geq \mu(a) \wedge \mu(a) = \mu(a)$ for all $a \in G$. By induction on n, it follows that $\mu(a^n) \geq \mu(e)$ for all $a \in G$ and for all positive integers n and also for $n = 0$.

Theorem 4.4 *Let G be a group and $\mu : G \to I$ be a fuzzy subset. Then μ is a fuzzy subgroup $\Leftrightarrow \mu(ab^{-1}) \geq \mu(a) \wedge \mu(b^{-1})$.*

Proof.

By Proposition 4.2, $\mu(a^{-1}) = \mu(a)$ and $\mu(ab^{-1}) \geq \mu(a) \wedge \mu(b^{-1}) = \mu(a) \wedge \mu(b)$.

Conversely,

1. $\mu(ea^{-1}) = \mu(a^{-1})$, *by Proposition* 4.2(2), $\mu(a^{-1}) \geq \mu(a)$.

2.$\mu(ab) = \mu(a(b^{-1})^{-1}) \geq \mu(a) \wedge \mu(b^{-1}) \geq \mu(a) \wedge \mu(b)$ since $\mu(b^{-1}) \geq \mu(b)$. □

For G abelian, we can use the additive notation and the fuzzy subgroups of (G, +) can be rewritten as $\mu(a - b) \geq \mu(a) \wedge \mu(b)$.

4.1 Level subgroups

In this Subsection, we define the level of subgroups and use them to study fuzzy subgroups of G. In this Subsection, α - cut becomes t - cut for convinience.

Definition 4.5 *Let G be a group and $\mu : G \to I$ be a fuzzy subgroup of G. The subgroup $\mu_t, t \in [0,1]$ and $t, s \leq \mu(e)$ is called a level subgroup of μ with respect to t.*

The following theorem clarifies that the level subgroups are not all distinct and therefore every level subgroup is indeed a subgroup of G.

Theorem 4.6 *Let G be a group and $\mu : G \to I$ be a fuzzy subgroup of G. Two level subgroups $\mu_{t_1}, \mu_{t_2} (t_1 < t_2)$ of μ are equal $\Leftrightarrow$ there is no $x \in G$ such that $t_1 < \mu(x) < t_2$.*

Proof. *Let $\mu_{t_1} = \mu_{t_2}$. Suppose there exists $x \in G$ such that $t_1 < \mu(x) < t_2$ then $\mu_{t_2} \nsubseteq \mu_{t_2}$, since x belongs to μ_{t_1}, but not to μ_{t_2}, which contradicts the hypothesis.*

Conversely, suppose there is no $x \in G$ such that $t_1 < \mu(x) < t_2$, with t_1 and t_2 as above. Since $t_1 < t_2$, we have $\mu_{t_2} \leq \mu_{t_1}$. Let $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$ and hence $\mu(x) \geq t_2$, since $\mu(x)$ can not lie between t_1 and t_2. Therefore $x \in \mu_{t_2}$. So $\mu_{t_1} \leq \mu_{t_2}$. Thus $\mu_{t_1} = \mu_{t_2}$. □

4.2 Generation of fuzzy subgroups

Definition 4.7 *Let G be a group and μ a fuzzy subset of G, $\mu \neq 0$. The smallest fuzzy subgroup of G containing μ, denoted by $< \mu >$ is called the fuzzy subgroup of G generated by μ.*

Proposition 4.8

$$\mu^{\star}(x) = \sup_{t \leq \sup \mu} \{t | x \in < \mu >\}$$

is the smallest fuzzy subgroup of G containing μ. □

Therefore $\mu^{\star}$ is indeed a fuzzy subgroup generated by μ in G, that is $\mu^{\star} = < \mu >$. In particular if μ is a fuzzy point $\mu = a^{\lambda}$ generated by a^{λ} is denoted by $< a^{\lambda} >$.

4.3 Cyclic fuzzy subgroups

Definition 4.9 *Let G be a group and a^{λ} a fuzzy point in G. A fuzzy subgroup μ is cyclic in G if there exists a fuzzy point a^{λ} such that $\mu = < a^{\lambda} >$.*

Proposition 4.10 *(Makamba, [12])*
Let $\mu = < a^{\lambda} >$ and

$$\mu(x) = \begin{cases} \lambda, & x \in < a >, \\ 0, & x \notin < a >, \quad \forall x \in G. \end{cases} \tag{4.38}$$

Then $\mu = \nu$.
Proof. *Let $b^{\beta} \in \nu$ if $\nu(b) \geq \beta$. If $b = e$, then $e \in < a >$. Hence $\nu(e) \geq \lambda \geq \beta$. Now $\mu(a) \geq \lambda$ and $\mu(e) \geq \mu(a) \geq \lambda \geq \beta$. So $b^{\beta} = e^{\beta} \in \mu$.*

Suppose $b \neq e$. $\nu(b) \geq \beta > 0$. Hence $\nu(b) = \lambda \geq \beta$ and $b \in < a >$. So $b = a^m$ for some $m \in \mathbb{Z}$. Therefore, $\mu(b) \geq \mu(a) \geq \lambda \geq \beta$. So $b^{\beta} \in \mu$, $a \in < a >$. By induction of μ, $\mu \leq \nu$. Hence $\mu = \nu$. □

4.4 Fuzzy abelian subgroups

In this Subsection, we define fuzzy abelian subgroups and its properties.

Definition 4.11 *Let μ be a fuzzy subgroup of G. Let $H = \{x \in G | \mu(x) = \mu(e)\}$. Then μ is a fuzzy abelian if H is an abelian subgroup of G.*

The above definition is weak according to Makamba [12], he proposed a strong definition in the place of the one given above.

Definition 4.12 *Let μ be a fuzzy subgroup of G. μ is a fuzzy abelian if μ^t is abelian for all $t \in [0, \mu(e)]$.*

4.5 Lattice of fuzzy subgroups

By lattice we mean a "partially ordered set". This subsection provide necessary conditions to classify and illustrate the fuzzy subgroups of G on a group-lattice diagram.

4.6 Fuzzy substructures join

The join of a family of fuzzy subgroups of G constitute complete lattices under the ordering of fuzzy set inclusion and form a descending chain of fuzzy sets with its least member $F_6(G)$ of the class of all fuzzy normal subgroups. That $F_6(G)$ is a modular lattice, can be shown by construction of the fuzzy set $\mu \cup \eta \vee \mu \cup \eta$, but this has failed to indicate for $F_i(G), i \neq 1,5$, that each small class is a sublattice of the large class (see for more information [1]).

The fuzzy subgroups of G can be represented by maximal chains. This representation is a way of classifying fuzzy subgroups of G using membership values. The following example illustrates the classification of fuzzy subgroups of G on the maximal chain. We have

$$\{e\} \subset (H_1^{\lambda_1}) \subset (H_2^{\lambda_2}) \subset \cdots \subset (H_{n-1}^{\lambda_{n-1}}) \subset (H_n^{\lambda_n}) \subset G, \tag{4.39}$$

where $H_i's$ are subgroups of G, the $\lambda_i's$ are values between [0,1] where $i = 0,\ 1,\ \cdots, n$.

4.7 Equivalence class of Fuzzy subgroups of G

In this Subsection, we discuss about the equivalence classes of fuzzy subgroups. We use the length of the maximal chain of the subgroups to determine the equivalence classes. The proposition below provides an algebraic formulation to determine the number of distinct equivalence classes of fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$.

Proposition 4.13 *(Murali and Makamba, [15])*
For any $n,m \in \mathbb{N}$, there are $2^{n+m+1}\sum_{r=0}^{m} \frac{1}{2^{-r}} \binom{n}{n-r} \binom{m}{r} - 1$ where $m \leq n$ distinct equivalence classes of fuzzy subgroups on $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$; where p and q are distinct.

The example below demonstrate that some keychains on distinct maximal chains determine the same equivalent class of fuzzy subgroups.

Example 4.14 *Let us take the following keychains $1 \quad \lambda \quad \lambda \qquad \gamma$ where $(1 > \lambda > \beta > \gamma > 0)$ on the follwoing two maximal chains*

$$o \subset p \subset pq \subset p^2q \subset p^2q^2$$
$$o \subset q \subset pq \subset pq^2 \subset p^2q^2$$

determine the same fuzzy subgroup whose pinned-flag is given by either

$$o^1 \subset p^\lambda \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma,$$

or

$$o^1 \subset q^\lambda \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma,$$

which can be reduced to

$$o^1 \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma.$$

4.8 Classification of Subgroups

Classification is a process of grouping items of certain similar characters together. In demonstrating a way of classifying subgroups we use the notion of maximal chains as follow. Consider a maximal chain

$$M_\mu : \mathbb{Z}_{p^n} \supset \mathbb{Z}_{p^{n-1}} \supset \cdots \supset \mathbb{Z}_p \supset 0. \tag{4.40}$$

M_μ defines a fuzzy subgroups μ of G as follows:

μ assumes λ_n on $\mathbb{Z}_{p^n} \setminus \mathbb{Z}_{p^{n-1}}$, λ_{n-1} on $\mathbb{Z}_{p^{n-1}} \setminus \mathbb{Z}_{p^{n-2}} \cdots$, λ_1 on $\mathbb{Z}_p \setminus \{0\}$ and 1 on $\{0\}$, where

$$1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0. \tag{4.41}$$

If the maximal chain in equation (4.41) has a length of four components, then we have 15 distinct equivalence classes of fuzzy subgroups on $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ for a specific m and n.

5 Disussion

The paper discuss about fuzzy sets, fuzzy subsets and fuzzy subgroups of a finite abelian group. We consider the implications of equivalence relation on fuzzy sets and fuzzy subgroups. In essence we study the fuzzy subgroups of finite abelian group. To be specific we take a cyclic group

$$G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m} \tag{5.42}$$

where p and q are two distinct prime, while n and m are fixed positive integers. G is a cyclic group of rank 2 and it has an order $u = p^n q^m$. We can determine the subgroups of G if n and m are known. Since G is a finite cyclic group of order o(G) = $p^n q^m$, for every divisor d of o(G), there is a unique subgroup of G of order d by a proposition in [11], page 92. Now clearly there are $(n+1)(m+1)$ many subgroups of G. So, in general

$$G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}, \tag{5.43}$$

has $(n+1)(m+1) = nm+n+m+1$ subgroups.

The subgroups of a group G are studies by using the notion of equivalence relations. Using equivalence relation, we classify fuzzy subgroups of finite abelian group in some special way. In fact we use equivalence relations to study the equivalence of fuzzy subgroups of G. The group structures can be classified by assigning equivalence classes to its fuzzy subgroups. The collection of the classes in these relation can be ordered. Furthermore the fuzzy subsets of X are characterized through the framework of flags and keychains. The distinct fuzzy subsets of X are obtained by interchanging the pins $\overline{\beta}$'s of a flag. The pins are allocated with their positions and the length of the chain is equal to the number of positions available in an n - chain which in our case is $n+1$. The length of the maximal chain of the subgroups of G is used to determine the equivalence classes of fuzzy subgroups. When

$$G = \mathbb{Z}_{p^1} + \cdots + \mathbb{Z}_{p^n}, \tag{5.44}$$

G has a length equal to $n+1$. Hence the number of equivalence classes of fuzzy subgroups is $2^{n+1} - 1$.

6 Concluding Remarks

In this article we discuss the classification of fuzzy subgroups of a finite abelian group G. We look at possible methods of classifying subgroups of a group. In particular, we look at the implications of equivalence relations on fuzzy sets and extend their applicability to the notion of fuzzy subgroups. These equivalence relations provide settings for classifying the fuzzy subgroups of G. The group structures can be classified by assigning equivalence classes to its fuzzy subgroups. This conditions under which the equivalence relation of fuzzy sets is equivalently described by their level sets is an alternative method of classification. Level subgroups characterized several properties of fuzzy subgroups in terms of level subgroups of G. For instance fuzzy subgroups were represented in the form of a chain of level subgroups. The collection of classes in this relation can be ordered on the subgroup - lattice.

Also the relationship between equivalent fuzzy subsets and equivalent fuzzy points establish the number of distinct keychains relative to equivalence of fuzzy points. We take a chain of fuzzy subsets by specifying a crisp point from the chain and obtain distinct keychains that are being hosted by the crisp point. We construct a fuzzy subgroups using lattice diagrams and form a subgroup - lattice of fuzzy subgroups of finite abelian group G = $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ and show how these satisfy the group structures together with their equivalence relations. This lead to the formation of various types of lattices and sublattices of fuzzy substructures of a group. The lattice will have more influence on further studies of fuzzy subgroups of finite abelian groups.

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