On the smoothness and the totally strong properties for nearness frames

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Abstract

In this paper we briefly explore the properties of smooth and totally strong nearness frames. It turns out that there is a relationship between them, in particular, totally strong nearness frames are smooth. We also show that the category of smooth nearness frames is coproductive, and that, as is the case with that of the totally strong ones, it is closed under completions.

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1 Preliminaries

Recall that a *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

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holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by 1_L and 0_L respectively, dropping the subscript L if it is clear from the context.

For a general theory of frames we refer to [7], and for nearness frames we refer to [1], [2] and [3]. In this introductory section we collect a few facts that will be relevant for our discussion, and fix notation.

An element a of a frame L is said to be *rather below* an element b, written $a \prec b$, if there is an element s such that $a \wedge s = 0$ and $s \vee b = 1$. Further, a is said to be *completely below* b, written $a \prec\!\prec b$, if there exists a doubly indexed sequence (x_{nk}), known as a scale between a and b, with $n = 0, 1, \dots$ and $k = 0, 1, \dots, 2^n$ such that

$$a = x_{n0},\ \ x_{nk} \prec x_{n\,k+1},\ \ x_{n\,2^n} = b, \text{ and } x_{nk} = x_{n+1\,2k},$$

for all $n = 0, 1, \dots$ and $k = 0, 1, \dots, 2^n$. Call L *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\prec a\}$ for each $a \in L$.

The *pseudocomplement* of an element a is the element $a^* = \bigvee\{x \in L \mid x \wedge a = 0\}$. We say a is *complemented* if $a \vee a^* = 1$.

A *frame homomorphism* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. A homomorphism is called *dense* if it maps only the bottom element to the bottom element. Associated with a homomorphism $h : L \to M$ is its *right adjoint* $h_* : M \to L$ given by

$$h_*(a) = \bigvee\{x \in L \mid h(x) \leq a\}.$$

A *cover* of a frame is a subset with join equal to the top. The set of all covers of L is denoted by $\mathrm{Cov}\,L$. For covers C and D of L, C is said to *refine* D, written $C \leq D$, if for every $c \in C$ there exists $d \in D$ such that $c \leq d$. For any $A \in \mathrm{Cov}\,L$ and $x \in L$, the element Ax of L is defined by $Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}$. If $A, B \in \mathrm{Cov}\,L$, then AB is the cover given by $AB = \{Ab \mid b \in B\}$. A cover A *star-refines* a cover B, written $A \leq^* B$, if $AA \leq B$.

For any $\mathfrak{N} \subseteq \mathrm{Cov}\,L$, the relation $\lhd_{\mathfrak{N}}$ (or simply $\lhd$) on L is defined by

$$x \lhd y \ \text{ if } \ Cx \leq y \ \text{ for some } \ C \in \mathfrak{N},$$

and $\mathfrak{N}$ is said to be *admissible* if $a = \bigvee\{x \in L \mid x \lhd a\}$ for each $a \in L$. If $a \lhd b$, we say a is *uniformly below* b. A *nearness* on L is an admissible filter $\mathfrak{N}$ in $(\mathrm{Cov}\,L, \leq)$. A *nearness frame* is a pair $(L, \mathfrak{N})$ where $\mathfrak{N}$ is a nearness on L. A frame has a nearness if and only if it is regular. Therefore all frames considered here are assumed to be regular.

We shall frequently abuse notation and denote a nearness frame by its underlying frame. If we have not named a nearness in question when talking about a nearness frame L, we shall, at times, write $\mathfrak{N}L$ for the nearness. Given a nearness frame L, the covers in $\mathfrak{N}L$ are called *uniform covers* of L. If L is a nearness frame and $C \in \mathrm{Cov}\,L$, $\check{C}$ is the cover defined by

$$\check{C} = \{x \in L \mid x \lhd c \text{ for some } c \in C\}.$$

A nearness frame L is said to be:

(1) *strong* if for every uniform cover C, $\check{C}$ is also a uniform cover.

(2) *fine* if $\mathfrak{N}L = \mathrm{Cov}\,L$.

Let L and M be nearness frames. A homomorphism $h\colon L \to M$ between the respective underlying frames is said to be:

(1) *uniform* if $h[C] \in \mathfrak{N}M$ for each $C \in \mathfrak{N}L$.

(2) a *surjection*, or *quotient map*, if it is onto and $\mathfrak{N}M = \{h[C] \mid C \in \mathfrak{N}L\}$. In this case we may refer to M as a quotient of L.

(3) a *strict surjection* if it is a dense surjection and the uniform covers $h_*[C], C \in \mathfrak{N}M$, generate $\mathfrak{N}L$.

We shall need the following results which appear in [1].

Lemma 1.1. *(i) If $a \lhd b$ in L, then $h(a) \lhd h(b)$ in M.*

(ii) If h is a dense surjection, then $a \lhd b$ in L implies $h_*h(a) \leq b$.

(iii) If h is a strict surjection, then $x \lhd y$ in M iff $h_*(x) \lhd h_*(y)$ in L.

(iv) If h is a strict surjection, then for any $a \in L$ and any $x \in M$, we have $a \lhd h_*(x)$ in L iff $h(a) \lhd x$ in M.

The category of nearness frames and uniform homomorphisms is denoted by **NFrm**. Its subcategory consisting of strong nearness frames will here be denoted by **StrNFrm**.

A nearness frame L is *complete* if any strict surjection $M \to L$ is an isomorphism, and a completion of L is any strict surjection $M \to L$ with complete M. Fine nearness frames are complete. Any nearness frame L has a completion $\gamma_L \colon CL \to L$, which is unique up to isomorphism.

In [4] a nearness frame is called *quotient-fine* if it is a quotient of a fine nearness frame. It is then shown that a nearness frame is quotient-fine if and only if its completion is fine.

In [5] a nearness frame L is called *interpolative* or has the *interpolation property* if, for every $a, b \in L$, $a \lhd b$ implies $a \lhd c \lhd b$ for some $c \in L$.

An *almost uniform* nearness frame is one which is strong and interpolative. We denote the category of almost uniform nearness frames by **AuNFrm**.

We also recall from [9] how coproducts are constructed in **NFrm**. Given a system $\{L_i\}_{i\in I}$ of nearness frames, let $\oplus_i L_i$ be the coproduct of the underlying frames, and

$$\iota_i : L_i \longrightarrow \oplus_i L_i$$

the coproduct injections. The frame $\oplus_i L_i$ is generated by elements of the form

$$\oplus_i a_i = \bigwedge_i \iota_i(a_i),$$

where the $a_i \in L_i$ are such that only finitely many of them are not equal to 1. The results in the following lemma appear in [9].

Lemma 1.2. *The elements $\oplus_i a_i$ have the following properties:*

(i) $\oplus_i a_i = 0$ iff $a_i = 0$ for some $i \in I$.

(ii) $0 \neq \oplus_i a_i \leq \oplus_i b_i$ *iff for all* $i \in I$, $a_i \leq b_i$.

(iii) $0 \neq \oplus_i a_i \lhd \oplus_i b_i$ *in* $\oplus_i L_i$ *iff for all* $i \in I$, $0 \neq a_i \lhd b_i$ *in* L_i.

Consider the covers

$$\oplus_i C_i = \{\oplus_i c_i \mid c_i \in C_i\}$$

where each C_i is a uniform cover of L_i, and only finitely many of the C_i are non-trivial, that is, unequal to $\{1\}$. The collection of all such $\oplus_i C_i$ generates a nearness on $\oplus_i L_i$; and the resulting nearness frame together with the coproduct injections $\iota_j \colon L_j \to \oplus_i L_i$ constitutes a coproduct in the category **NFrm**.

2 Smooth nearness frames

Smooth nearness frames were introduced in [2] as an ad-hoc means to studying completion in nearness frames. In this section we investigate some properties of these nearness frames, culminating in showing that the smooth property is not changed under completions.

Call a nearness frame (L, μ) *smooth* if for each uniform cover C, the set

$$C^s = \{x \in L \mid x^{**} \leq y \text{ for some } y \in C\}$$

is also a uniform cover. Write **SmNFrm** for the category of smooth nearness frames, which, in consequence, is a full subcategory of **NFrm**

Remark 2.1. Note here that since $x \lhd y$ implies $x^{**} \leq y$, we have $\check{C} \subseteq C^s$, so that, as observed in [2],

Every strong nearness frame is smooth. However, a smooth nearness frame need not be strong.

Thus, **StrNFrm** $\subsetneqq$ **SmNFrm**.

It is clear that quotient-fine (and, hence, fine) nearness frames are smooth.

Let L be a nearness frame. If A is a uniform cover, then the set

$$A^{**} = \{x^{**} \in L \mid x \in A\}$$

is also a uniform cover (since A refines A^{**}).

The following characterization of smooth nearness frames follows naturally.

Lemma 2.2. *A nearness frame L is smooth iff for each $A \in \mathfrak{N}L$, there exists $B \in \mathfrak{N}L$ such that B^{**} refines A.*

Proof. $(\Rightarrow)$ Suppose L is smooth. Let A be a uniform cover. Then

$$A^s = \{x \in L \mid x^{**} \leq a, \text{ some } a \in A\}$$

is also a uniform cover. By the hypothesis, $(A^s)^{**}$ is a uniform cover (as observed above) of the desired kind refining A.

$(\Leftarrow)$ Conversely, assume that the condition holds. For $A \in \mathfrak{N}L$, let B^{**} refine A for some $B \in \mathfrak{N}L$. Then $B \subseteq A^s$, so that $B \leq A^s$. Consequently, $A^s \in \mathfrak{N}L$, so that L is smooth. □

Example 2.3. It is worth noting here that the cover A^{**}, as introduced above, is not necessarily the same as the set

$$A^r = \{x \in A \mid x = x^{**}\}$$

of all regular elements in A. As an example, let L be a nearness frame where the underlying frame is compact and non-Boolean. (Trivially, every nearness on a Boolean frame is smooth). Let A be a uniform cover of L consisting only of regular elements and let $x \in L$ be a non-regular element such that $x^{**} \notin A$. Put $B = A \cup \{x\}$. Then $B \in \mathfrak{N}L$ and $B^{**} = A \cup \{x^{**}\}$, but

$$B^r = \{y \in B \mid y = y^{**}\} = A.$$

In order to show that **SmNFrm** is coproductive, we shall need the following lemma.

Lemma 2.4. *Let $\oplus_i L_i$ be the coproduct of a family $\{L_i\}_{i\in I}$ of frames. Then for each element $\oplus_i a_i \in L$,*

$$(\oplus_i a_i)^{**} = \oplus_i (a_i^{**}).$$

Proof. Let $\iota_i : L_i \longrightarrow \oplus_i L_i$ be the ith coproduct injection. We first show that

$$(\dagger) \qquad (\oplus_i a_i)^* = \bigvee_i \iota_i(a_i^*).$$

By definition, for each index k, and for any $x \in L_k$, $\iota_k(x) = \oplus_i b_i$, where $b_k = x$ and $b_i = 1$ for $i \neq k$. Now if $\oplus_i c_i$ is any element of $\oplus_i L_i$ such that

$$(\oplus_i a_i) \wedge (\oplus_i c_i) = 0,$$

then

$$\oplus_i (a_i \wedge c_i) = 0,$$

so that $a_k \wedge c_k = 0$ for some index k. This implies $c_k \leq a_k^*$. Consequently

$$\oplus_i c_i \leq \iota_k(a_k^*) \leq \bigvee_i \iota_i(a_i^*).$$

Since the elements $\oplus_i x_i$ generate the frame $\oplus_i L_i$, it follows that if any element of $\oplus_i L_i$ does not meet $\oplus_i a_i$, then it is below $\bigvee_i \iota_i(a_i^*)$. Therefore

$$(\oplus_i a_i)^* \leq \bigvee_i \iota_i(a_i^*).$$

But, by applying the infinite distributive law,

$$(\oplus_i a_i) \wedge \left(\bigvee_i \iota_i(a_i^*) \right) = 0.$$

Therefore $\bigvee_i \iota_i(a_i^*)$ is the largest element of $\oplus_i L_i$ disjoint from $\oplus_i a_i$. Thus, $(\dagger)$ holds.

Second, we apply $(\dagger)$ and the fact that for each index k,

$$(\iota_k(x))^* = \iota_k(x^*),$$

to obtain

$$(\oplus_i a_i)^{**} = \left(\bigvee_i \iota_i(a_i^*) \right)^* = \bigwedge_i (\iota_i(a_i^*))^* = \bigwedge_i \iota_i(a_i^{**}) = \oplus_i (a_i^{**}),$$

establishing the desired result. □

Proposition 2.5. **SmNFrm** *is coproductive in* **NFrm***.*

Proof. Let $\{L_i\}_{i\in I}$ be a family of smooth nearness frames. We show that their coproduct $\oplus_i L_i$ is also smooth. To see that, let A be a uniform cover in the coproduct, and let $\oplus_i A_i$ be a refinement of A (where each $A_i \in \mathfrak{N}L_i$). Let $A_{i_1},\ldots,A_{i_m}$ be the nontrivial covers among the covers A_i's.

We construct a uniform cover of the form B^{**} (in the coproduct nearness) refining A as follows: For each $i \in \{i_1,\ldots,i_m\}$, let $B_i^{**} \in \mathfrak{N}L_i$ refine A_i. We let $B_i^{**} = \{1\}$ for the other i's. Then, making use of the above lemma,

$$B^{**} = (\oplus_i B_i)^{**} = \oplus_i B_i^{**}$$

refines $\oplus_i A_i$ which refines A. Hence the desired result follows. □

In [2] it is shown that any dense surjection $h : L \longrightarrow M$ with L smooth is in fact a strict surjection, and consequently any weak completion $h : L \longrightarrow M$, where L is smooth, becomes a completion. Here we show the following result.

Proposition 2.6. *If $h : L \longrightarrow M$ is a dense surjection, then L is smooth iff M is smooth.*

Proof. ($\Rightarrow$) Suppose L is smooth. Let $C \in \mathfrak{N}M$. To show that M is smooth, we need $D \in \mathfrak{N}M$ such that $D^{**} \leq C$. Since $h_*[C] \in \mathfrak{N}L$, there exists $B \in \mathfrak{N}L$ such that $B^{**} \leq h_*[C]$, since L is smooth. Since h preserves pseudocomplements, being a dense onto map, we have $h[B^{**}] = h[B]^{**}$. Thus, $h[B]$ is a uniform cover of M such that $h[B]^{**}$ refines C, and therefore M is smooth.

($\Leftarrow$) Suppose M is smooth. Let A be a uniform cover of L. Since h is a strict surjection, there is a uniform cover B of M such that $h_*[B] \leq A$. Since M is smooth, by the hypothesis, B^{**} is a uniform cover of M. Therefore $h_*[B^{**}]$ is a uniform cover of L since h is a strict surjection. Since h is a dense onto homomorphism, h_* commutes with pseudocomplementation; so that

$$h_*(b^{**}) = h_*(b)^{**}$$

for each $b \in B$, and hence $h_*[B^{**}] = h_*[B]^{**}$. But now $h_*[B]^{**}$ refines A^{**}; therefore A^{**} is also a uniform cover, and hence L is smooth. □

The following corollary is evident from the above result, since the completion map is a strict surjection.

Corollary 2.7. *A nearness frame is smooth iff its completion is smooth.*

3 Totally strong nearness frames

By imposing a stronger refinement ordering on uniform covers, in particular, one which uses scales in the manner in which the completely below relation is defined, we introduce, in this section, a type of nearness frames called the totally strong ones and establish that their category, namely **TStrNFrm**, is closed under completions, and that the inclusions **AuNFrm** $\subseteq$ **TStrNFrm** $\subseteq$ **StrNFrm** hold.

Definition 3.1. Let L be a nearness frame, and $A, B \in \mathfrak{N}L$. Write $A \lhd\!\lhd_s B$ if there is an interpolating sequence of uniform covers (C_{nk}) between A and B, where

$$C_{00} = A,\ C_{01} = B,\ \ C_{nk} = C_{n+1\,2k},\ \text{and}\ C_{nk} \lhd C_{n\,k+1}$$

for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$. In this case we say *A scale refines B*. We call a nearness frame *totally strong* if every uniform cover is scale refined by a uniform cover.

Clearly, if $A \lhd\!\lhd_s B$, then $A \lhd B$. Consequently, every totally strong nearness frame is strong. We write **TStrNFrm** for the category of totally strong nearness frames. Thus, **TStrNFrm** $\subseteq$ **StrNFrm**, and, since **StrNFrm** $\subseteq$ **SmNFrm**, every totally strong nearness frame is smooth.

In order to show that every almost uniform nearness frame is totally strong, we need the following result which shows that interpolation in the underlying frame L is transferred to its nearness $\mathfrak{N}L$.

Lemma 3.2. *Suppose L is an interpolative nearness frame, and suppose $A, B \in \mathfrak{N}L$ with $A \lhd B$. Then there exists $C \in \mathfrak{N}L$ such that $A \lhd C \lhd B$.*

Proof. Let $A, B \in \mathfrak{N}L$ be such that $A \lhd B$. Then for each $a \in A$, there exists $b_a \in B$ such that $a \lhd b_a$. Since L is interpolative, there exists $c_a \in L$ such that $a \lhd c_a \lhd b_a$. Form the set

$$C = \{c_a \in L \mid a \in A\}.$$

Then C is a uniform cover, since A refines it. Furthermore $A \lhd C \lhd B$ by the way C is constructed. □

Proposition 3.3. *If L is almost uniform, then it is totally strong.*

Proof. Let $B \in \mathfrak{N}L$. Since L is strong, there exists $A \in \mathfrak{N}L$ such that $A \lhd B$. By Lemma 3.2, $\lhd$ interpolates in $\mathfrak{N}L$, since L is interpolative. Therefore $A \lhd\lhd_s B$, since, by the axiom of countable dependent choice, a scale of uniform covers witnessing this can be constructed in the same manner as done in [8, Lemma 1.5]. □

As an observation from the above two results, it should be evident that if L is a strong nearness frame with the property that whenever $A \lhd B$ in $\mathfrak{N}L$, there exists $C \in \mathfrak{N}L$ such that $A \lhd C \lhd B$, then L is totally strong.

In our next set of results we aim to show that the category **TStrNFrm** is closed under completions. Our proof will be facilitated by noting the following: if $h : L \to M$ is a uniform frame homomorphism and A scale refines B in L, then $h[A]$ scale refines $h[B]$ in M, for if (C_{nk}) is a scale of uniform covers of L witnessing $A \lhd\lhd_s B$, then clearly $(h[C_{nk}])$ is a scale of uniform covers of M witnessing $h[A] \lhd\lhd_s h[B]$. On the other hand, if h is a strict surjection and U scale refines V in M, then $h_*[U]$ scale refines $h_*[V]$ in L, for if (W_{nk}) is a scale of uniform covers of M witnessing $U \lhd\lhd_s V$, then, by the strictness of h, $(h_*[W_{nk}])$ is a scale of uniform covers of L witnessing $h_*[U] \lhd\lhd_s h_*[V]$.

Lemma 3.4. *Let $h : L \longrightarrow M$ be a strict surjection. Then L is totally strong iff M is totally strong.*

Proof. $(\Rightarrow)$ Suppose L is totally strong and let U be a uniform cover of M. Then, by strictness, $h_*[U]$ is a uniform cover of L. Since L is totally strong, there is a uniform cover A of L that scale refines $h_*[U]$. By what we have observed above, $h[A]$ is a uniform cover of M scale refining $h[h_*[U]] = U$. Therefore M is totally strong.

($\Leftarrow$) Conversely, suppose M is totally strong and let A be a uniform cover of L. By strictness, there is a uniform cover U of M such that $h_*[U] \leq A$. Since M is totally strong, there is a uniform cover V of M which scale refines U. Then $h_*[V]$ is a uniform cover of L scale refining $h_*[U]$, and hence scale refining A. Therefore L is totally strong. □

Since completion maps are strict surjections, we deduce the following result.

Corollary 3.5. *A nearness frame is totally strong if and only if its completion is totally strong.*

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